Higher School in Applied Sciences
Preparatory Department
Course: Algebra 1
Academic Year: 2023/2024

Set of exercises 2

**Exercise 1.** Let $\mathcal{G}_n$ the subset of $\mathbb{N}$ defined as $\mathcal{G}_n = \{0, 1, 2, 3, \cdots, n-1\}$ on which we define the law of composition:

$\forall x, y \in \mathcal{G}_n,\ x \oplus y = s$ with $s:$ the rest of the euclidean division of $(x + y)$ by $n$.

1. Verify that $(\mathcal{G}_n, \oplus)$ is an abelian group.
2. We choose $n = 10$
   (a) Calculate $4 \oplus 8; \quad 3 \oplus (-7)$
   (b) Resolve in $\mathcal{Z}_{10}$ the equation: $x \oplus 3 = 1$

**Exercise 2.** Let $(E, *)$ be a group. We define the center of $E$:

$$Z(E) = \{x \in E : \forall y \in E,\ x * y = y * x\}$$

1. Show that $Z(E)$ is a subgroup of $(E, *)$
2. Is $Z(E)$ abelian?

**Exercise 3.** Let $(\mathbb{R} \setminus \{-3\}, *)$ be a group with $*$ defined as: $a * b = ab + 3(a + b + 2)$

Let $f : (\mathbb{R} \setminus \{-3\}, *) \to (\mathbb{R}^*, .)$
$x \mapsto x + 3$

1. Show that $f$ is a group homomorphism.
2. Calculate using two different ways the identity element $e$ and the inverse $x^{-1}$.

**Exercise 4.** Let $f : (E, *) \to (F, \Delta)$ be a group homomorphism.

We define the kernel of $f$ as:

$ker(f) = \{x \in E : f(x) = e_2\}$ with $e_2$ is the identity element of $(F, \Delta)$.

1. Prove that $ker(f)$ is a subgroup of $(E, *)$.
2. Show that: $f$ injective $\Leftrightarrow ker(f) = \{e_1\}$ with $e_1$ is the identity element of $(E, *)$.

**Exercise 5.** Let $\mathcal{G}_n$ the subset of $\mathbb{N}$ defined as $\mathcal{G}_n = \{0, 1, 2, 3, \cdots, n-1\}$ on which we define two laws of composition:

$\forall x, y \in \mathcal{Z}_n,\ x \oplus y = s$ with $s:$ the rest of the euclidean division of $(x+y)$ by $n$.
$x \otimes y = p$ with $p:$ the rest of the euclidean division of $xy$ by $n$

1. Verify that $(\mathcal{G}_n, \oplus, \otimes)$ is a commutative ring.
2. We choose $n = 10$
3. Calculate $2 \otimes 9; \quad 9^{-1} \oplus 5; \quad 3^{-1} \oplus 6; \quad 3^2; \quad 7^2$
4. Resolve in $\mathcal{Z}_{10}$ the following equations:

   $(3 \otimes x) \oplus 5 = 0$
   $x^2 \oplus 1 = 0$

**Exercise 6.** Let $(E, +, \times)$ be a ring.
We say that $x$ is nilpotent if $\exists n \in \mathbb{N}^*$ such that $x^n = 0$.
Prove the following statements:

1. $(x$ nilpotent$) \wedge (xy = yx) \Rightarrow xy$ nilpotent
2. $xy$ nilpotent $\Rightarrow yx$ nilpotent

**Exercise 7.** we define $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} / a, b \in \mathbb{Q}\}$

1. Show that $\left(\mathbb{Q}[\sqrt{2}], +, \times\right)$ is a subring of $(\mathbb{R}, +, \times)$.
2. Deduce that $\left(\mathbb{Q}[\sqrt{2}], +, \times\right)$ is a field.