Higher School in Applied Sciences
Preparatory Department
Course: Algebra 1
Academic Year: 2023/2024

Set of exercises 1

Exercise 1. Using truth tables, prove that the following statements are logically equivalent

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) \tag{1}$$
$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \tag{2}$$
$$(P \vee Q) \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R) \tag{3}$$
$$\tag{4}$$

Exercise 2.

1. Are the following statements true or false?
   (a) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x + y > 0$
   (b) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x + y > 0$
   (c) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y^2 > x$
2. Give their negation

Exercise 3. Prove by contraposition that

$$xy \text{ even} \Rightarrow x \text{ even or } y \text{ even}$$

Exercise 4. Let $x \in \mathbb{R}$ such that $\forall \varepsilon > 0, x \leq \varepsilon$
Prove by contradiction that $x \leq 0$

Exercise 5.

1. Using mathematical induction prove that $5^n - 2^n$ is a multiple of 3.
2. is $x^n - y^n$ a multiple of $x - y$ for all $x, y \in \mathbb{N}$ with $x > y$?
3. For which natural numbers n is $n! > 3^n$?

Exercise 6. Let $E = \{a, b\}$

1. Give the set $\mathbb{P}(E)$
2. The following items are elements of which set?
   (a, b) , $\{a\}, (a, \{b\}), (\{a\}, \{b\}), \{\{a\}, \{b\}\}, (a, E)$

Exercise 7. let E be a set and $A, B \in \mathbb{P}(E)$
Resolve in $\mathcal{P}(E)$ the following equations:

1. $X \cap A = B$
2. $X \setminus A = B$

Exercise 8. Let E be a set and $A, B, C \in \mathcal{P}(E)$
We define the symmetric difference of A and B as:

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

1. Show that $A \Delta B = (A \setminus B) \cup (B \setminus A)$

2. Calculate $A\Delta A, \quad A\Delta\overline{A}, \quad A\Delta E, \quad A\Delta\emptyset$

3. Assuming that $A\Delta B = A\Delta C$, prove that $B = C$.

Exercise 9. The characteristic map is defined as

$$\begin{array}{rccl} \varphi_A: & E & \to & \{0,1\} \\ & x & \mapsto & \varphi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \end{array}$$

1. Prove that $\varphi_A = \varphi_B \Leftrightarrow A = B$

2. Establish the following properties:

   (a) $\varphi_{\overline{A}} = 1 - \varphi_A$

   (b) $\varphi_{A\cap B} = \varphi_A\varphi_B$

   (c) $\varphi_{A\cup B} = \varphi_A + \varphi_B - \varphi_A\varphi_B$

3. Deduce that $(A\cup B = A\cup C) \wedge (A\cap B = A\cap C) \Rightarrow (B = C)$

Exercise 10. 1. Knowing that $E = F = \mathbb{R}$, are the following injections? surjections?

$$f: \begin{array}{ccc} E & \to & F \\ x & \mapsto & \dfrac{x^2+1}{x^2-1} \end{array} \qquad g: \begin{array}{ccc} E & \to & F \\ x & \mapsto & exp\,(x+2) \end{array} \qquad h: \begin{array}{ccc} E & \to & F \\ x & \mapsto & \sqrt{1-x^2} \end{array}$$

2. Find E and F such that $f,\ g$ et h are bijections, and find their inverses.

3. Calculate $g([1,2]), g^{-1}(]1,3])$

Exercise 11. Let $f: E \to F$ and $A, B \in P(E)$. Prove that:

1. $f(A\cup B) = f(A)\cup f(B)$

2. f injective $\Rightarrow f(A\cap B) = f(A)\cap f(B)$

Exercise 12. Let E be a nonempty set and $A, B \in P(E)$.
We define the mapping f:
$$f: \begin{array}{ccc} P(E) & \to & P(A)\times P(B) \\ X & \mapsto & (X\cap A\,,\, X\cap B) \end{array}$$

1. Prove that f injective $\Leftrightarrow A\cup B = E$

2. Show that f surjective $\Leftrightarrow A\cap B = \emptyset$

3. At what condition is f bijective? give its inverse f^{-1}.