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ALGEBRA 1
LECTURE NOTES

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Contents

Chapter 1

Logic, sets and mappings

Logic

Definition 1. A statement (P) is a declarative sentence that is either true or false but not both.

Logical operators

Definition 2 (Negation). The negation of P denoted by $\neg P$ is the statement that says the opposite of P.

P	1	0
$\neg P$	0	1

Definition 3 (Conjonction). The conjonction of P and Q denoted by $P \wedge Q$ means (P) and (Q).

P	1	1	0	0
Q	1	0	0	1
$P \wedge Q$	1	0	0	0

Definition 4 (Disjonction). The disjonction of P and Q denoted by $P \vee Q$ means (P) or (Q)

P	1	1	0	0
Q	1	0	0	1
$P \vee Q$	1	1	0	1

Definition 5 (Conditional). The conditional statement or implication denoted by $P \Rightarrow Q$ reads "if P then Q" or "P implies Q". P is called the hypothesis and Q the result. The statement $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$.

P	1	1	0	0
Q	1	0	0	1
$P \Rightarrow Q$	1	0	1	1

Definition 6 (Biconditional). The biconditional statement denoted by $P \Leftrightarrow Q$ reads "P if and only if Q".

P	1	1	0	0
Q	1	0	0	1
$P \Leftrightarrow Q$	1	0	1	0

Definition 7 (Logical equivalency)**.** If the biconditional statement $P \Leftrightarrow Q$ is true, we say that P and Q are logically equivalent and we write $P \equiv Q$. In this case P and Q are both true or both false.

Theorem 1.

1. $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$ (Contrapositive)
2. $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$ (Negation of implication)
3. $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Theorem 2 (Morgan's Law)**.**

1. $\neg(P \wedge Q) = \neg P \vee \neg Q$
2. $\neg(P \vee Q) = \neg P \wedge \neg Q$

Quantifiers

Let U be a nonempty set

Definition 8. The universal quantifier denoted by $(\forall x \in U), (P(x))$ reads "the statement P holds for all values of x in U".

Definition 9. The existential quantifier denoted by $(\exists x \in U), (P(x))$ reads "the statement P holds for at least one value of x in U".

Theorem 3 (Negation of quantifiers)**.**

1. $\neg\Big[(\forall x \in U), (P(x))\Big] \equiv \Big[(\exists x \in U), \neg(P(x))\Big]$.
2. $\neg\Big[(\exists x \in U), (P(x))\Big] \equiv \Big[(\forall x \in U), \neg(P(x))\Big]$.

Reasoning methods

Proof by Contrapositive

We know from theorem 1 that: $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$

Example 1. Let us prove that:

$$\forall n \in \mathbb{Z} \quad [n^2 - 6n + 5] \text{ even} \Rightarrow n \text{ odd}$$

It is simpler to prove the contrapositive:

$$\forall n \in \mathbb{Z} \quad n \text{ even} \Rightarrow [n^2 - 6n + 5] \text{ odd}$$

Let $n \in \mathbb{Z}$

$$\begin{aligned} n \text{ even} \quad &\Rightarrow \quad n = 2k \text{ with } k \in \mathbb{Z} \\ &\Rightarrow \quad n^2 - 6n + 5 = (2k)^2 - 6(2k) + 5 \\ &\qquad = 2(2k^2 - 6k + 2) + 1 \\ &\qquad = 2k' + 1 \end{aligned}$$

We conclude that $n^2 - 6n + 5$ is odd.

Proof by Contradiction

To prove that a statement P is true by contradiction, we assume that $\neg P$ is true and we must find some contradiction.

Example 2. Let us prove by contradiction that forall prime numbers p, $\sqrt{p}$ is irrational. Suppose that:

$$\sqrt{p} \in \mathbb{Q} \Rightarrow \sqrt{p} = \frac{a}{b} \text{ with } GCD(a,b) = 1 \Rightarrow p = \frac{a^2}{b^2} \Rightarrow a^2 = pb^2$$

$$p|a^2 \Rightarrow p|a \Rightarrow a = pk \Rightarrow a^2 = p^2k^2 = pb^2 \Rightarrow b^2 = pk^2$$

$$p|b^2 \Rightarrow p|b \Rightarrow p|GCD(a,b) \Rightarrow p|1 \text{ (logical contradiction)}$$

We know from theorem 1 that: $\neg(P \Rightarrow Q) \equiv \left[P \wedge (\neg Q)\right]$

Example 3. Let us prove by contradiction that:

$$(\text{forall prime numbers p, } \sqrt{p} \text{ is irrational }) \Rightarrow \left(\sqrt{2} + \sqrt{5}\right) \text{ is irrational}$$

Suppose that $(\sqrt{2} + \sqrt{5}) \in \mathbb{Q} \Rightarrow \sqrt{2} + \sqrt{5} = \frac{a}{b}$ with $gcd(a,b) = 1 \Rightarrow \sqrt{5} = \frac{a}{b} - \sqrt{2} \Rightarrow 5 = \left(\frac{a}{b} - \sqrt{2}\right)^2$
$\Rightarrow 5 = \frac{a^2}{b^2} + 2 - 2\frac{a\sqrt{2}}{b} \Rightarrow \sqrt{2} = \left(3 - \frac{a^2}{b^2}\right) \times \frac{b}{-2a} = \frac{3b^2 - a^2}{-2ba} \Rightarrow \sqrt{2} \in \mathbb{Q}$ which contradicts our assumption.

Proof by Induction

Let $n_0 \in \mathbb{N}$

To prove that $(\forall n \geq n_0), (P(n))$ is true by induction, we must

1. Prove that $P(n_0)$ is true.
2. Suppose $P(n)$ holds true for some value $n > n_0$.
3. Prove that $P(n+1)$ is true.

Example 4. Let us prove by induction that:

$$\forall n \geq 1 \quad 1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

1. We prove that $P(n)$ is true for $n = 1$

$$\frac{1(1+1)}{2} = 1$$

2. Suppose that $P(n)$ is true for some $n \geq 1$, ie: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3. We prove that $P(n+1)$ is true i.e: $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\sum_{k=1}^{n+1} k = (1 + 2 + 3 + \cdots + n) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$