Rational fractions

Definition 48. A rational fraction is any expression of the form: $F = \dfrac{P}{Q}$
with $P, Q \in K[X]$ and $Q \neq 0$. The set of rational fractions is denoted $K(X)$.
$deg(F) = deg(P) - deg(Q)$
if $deg(F) < 0$, F is said to be proper, otherwise F is called improper.

Example 45. $F(X) = \dfrac{X^2 - 3X + 2}{X^3 - 2X + 1} \Rightarrow deg(F) = 2 - 3 = -1$

Definition 49. Let $F \in K(X)$ be a rational fraction. We call the irreducible form of F the pair (A, B) with $F = \dfrac{A}{B}$ and $GCD(A, B) = 1$.

Example 46. $F(X) = \dfrac{X^2 - 3X + 2}{X^3 - 2X + 1} = \dfrac{P(X)}{Q(X)}$
We are looking for the common roots: $\alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha_1 = 1$ and $\alpha_2 = 2$
$Q(1) = 0$ and $Q(2) \neq 0 \Rightarrow (X - 1)$ divides P and Q
$\Rightarrow F(X) = \dfrac{(X-1)(X-2)}{(X-1)(X^2+X-1)} = \dfrac{X-2}{X^2+X-1} = \dfrac{A(X)}{B(X)}$ with $GCD(A, B) = 1$

Partial-Fraction Decomposition in $\mathbb{R}(X)$

Definition 50. A partial fraction in $\mathbb{R}(X)$ is one of the following:

- Any monomial of $\mathbb{R}[X]$.
- Any rational fraction of the form:

$$\frac{k}{(X-\alpha)^p} \text{ with } k, \alpha \in \mathbb{R} \text{ and } p \in \mathbb{N}$$

- Any rational fraction of the form:

$$\frac{aX+b}{(X^2+cX+d)^p} \text{ with } k, \alpha \in \mathbb{R}, p \in \mathbb{N}^* \text{ and } c^2 - 4d < 0$$

Theorem 18. Any rational fraction is uniquely written as the sum of partial fractions of $\mathbb{R}(X)$.

Example 47. Decompose $F(X) = \dfrac{X^3}{X^2 - X - 2}$ into partial fractions in $\mathbb{R}(X)$

$$\begin{array}{rrrr|l}
X^3 & & & & X^2 - X - 2 \\
X^3 & -X^2 & -2X & & X + 1 \\
\hline
 & X^2 & +2X & & \\
 & X^2 & -X & -2 & \\
\hline
 & & 3X & +2 &
\end{array}$$

$$X^3 = (X+1)(X^2-X-2) + (3X+2) \Rightarrow \frac{X^3}{X^2-X-2} = \frac{(X+1)(X^2-X-2)+3X+2}{X^2-X-2}$$
$$= X + 1 + \frac{3X+2}{X^2-X-2}$$

$X^2 - X - 2 = (X+1)(X-2)$

$\Rightarrow \dfrac{3X+2}{X^2-X-2} = \dfrac{3X+2}{(X+1)(X-2)} = \dfrac{k_1}{X+1} + \dfrac{k_2}{X-2}$

Calculation of k_1

$\dfrac{3X+2}{(X+1)(X-2)}(X+1) = \dfrac{k_1}{X+1}(X+1) + \dfrac{k_2}{X-2}(X+1)$

$\Rightarrow \dfrac{3X+2}{X-2} = k_1 + \dfrac{k_2(X+1)}{X-2}$

$X = -1 \Rightarrow \dfrac{3(-1)+2}{(-1)-2} = k_1 + \dfrac{k_2((-1)+1)}{(-1)-2} \Rightarrow \boxed{k_1 = \dfrac{1}{3}}$

Calculation of k_2

$\dfrac{3X+2}{(X+1)(X-2)}(X-2) = \dfrac{k_1}{X+1}(X-2) + \dfrac{k_2}{X-2}(X-2)$

$\Rightarrow \dfrac{3X+2}{X+1} = \dfrac{k_1(X-2)}{X+1} + k_2$

$X = 2 \Rightarrow \dfrac{3(2)+2}{2+1} = \dfrac{k_1(2-2)}{2+1} + k_2 \Rightarrow \boxed{k_2 = \dfrac{8}{3}}$

Conclusion:

$$\boxed{\frac{X^3}{X^2-X-2} = X + 1 + \frac{1}{3(X+1)} + \frac{8}{3(X-2)}}$$

Example 48. Decompose $F(X) = \dfrac{X^2}{X^3 - X^2 + X - 1}$ into partial fractions in $\mathbb{R}(X)$

$X^3 - X^2 + X - 1 = (X^2+1)(X-1) \Rightarrow F = \dfrac{X^2}{(X^2+1)(X-1)} = \dfrac{aX+b}{X^2+1} + \dfrac{k_1}{X-1}$

Calculation of k_1

$\dfrac{X^2}{(X^2+1)(X-1)}(X-1) = \dfrac{aX+b}{X^2+1}(X-1) + \dfrac{k_1}{X-1}(X-1)$

$\Rightarrow \dfrac{X^2}{X^2+1} = \dfrac{aX+b}{X^2+1}(X-1) + k_1$

$X = 1 \Rightarrow \dfrac{1^2}{1^2+1} = \dfrac{a+b}{1^2+1}(1-1) + k_1 \Rightarrow \boxed{k_1 = \dfrac{1}{2}}$

Calculation of a, b

$\dfrac{X^2}{(X^2+1)(X-1)}(X^2+1) = \dfrac{aX+b}{X^2+1}(X^2+1) + \dfrac{k_1}{X-1}(X^2+1)$

$\Rightarrow \dfrac{X^2}{X-1} = (aX+b) + \dfrac{k_1}{X-1}(X^2+1)$

$X = i \Rightarrow \dfrac{i^2}{i-1} = (ai+b) + \dfrac{k_1}{i-1}(i^2+1) \Rightarrow \dfrac{1}{2} + \dfrac{1}{2}i = ai + b \Rightarrow a = b = \dfrac{1}{2}$

Conclusion:

$$\boxed{\frac{X^2}{(X^2+1)(X-1)} = \frac{\frac{1}{2}X + \frac{1}{2}}{X^2+1} + \frac{\frac{1}{2}}{X-1} = \frac{X+1}{2(X^2+1)} + \frac{1}{2(X-1)}}$$

Example 49. Decompose $F(X) = \dfrac{X^4+2}{X^2(X^2+1)^2}$ into partial fractions in $\mathbb{R}(X)$

$$F(X) = \frac{X^4+2}{X^2(X^2+1)^2} = \frac{k_1}{X} + \frac{k_2}{X^2} + \frac{a_1X+b_1}{X^2+1} + \frac{a_2X+b_2}{(X^2+1)^2}$$

Note that $F(X) = F(-X)$ with:

$$F(-X) = \frac{-k_1}{X} + \frac{k_2}{X^2} + \frac{-a_1X+b_1}{X^2+1} + \frac{-a_2X+b_2}{(X^2+1)^2}$$

By identification, $k_1 = -k_1 \quad a_1 = -a_1 \quad a_2 = -a_2 \Rightarrow k_1 = a_1 = a_2 = 0$

$$F(X) = \frac{k_2}{X^2} + \frac{b_1}{X^2+1} + \frac{b_2}{(X^2+1)^2}$$

Calculation of k_2

$$\frac{X^4+2}{X^2(X^2+1)^2}X^2 = \frac{k_2}{X^2}X^2 + \frac{b_1}{X^2+1}X^2 + \frac{b_2}{(X^2+1)^2}X^2$$

$$\Rightarrow \frac{X^4+2}{(X^2+1)^2} = k_2 + \frac{b_1}{X^2+1}X^2 + \frac{b_2}{(X^2+1)^2}X^2$$

$$X = 0 \Rightarrow \boxed{k_2 = 2}$$

Calculation of b_2

$$\frac{X^4+2}{X^2(X^2+1)^2}(X^2+1)^2 = \frac{k_2}{X^2}(X^2+1)^2 + \frac{b_1}{X^2+1}(X^2+1)^2 + \frac{b_2}{(X^2+1)^2}(X^2+1)^2$$

$$\Rightarrow \frac{X^4+2}{X^2} = \frac{k_2}{X^2}(X^2+1)^2 + b_1(X^2+1) + b_2$$

$$X = i \Rightarrow \frac{i^4+2}{i^2} = b_2 \Rightarrow \boxed{b_2 = -3}$$

$$\frac{X^4+2}{X^2(X^2+1)^2} = \frac{2}{X^2} + \frac{b_1}{X^2+1} - \frac{3}{(X^2+1)^2}$$

Calculation of b_1

$$X = 1 \Rightarrow \frac{1^4+2}{1^2(1^2+1)^2} = \frac{2}{1^2} + \frac{b_1}{1^2+1} - \frac{3}{(1^2+1)^2} \Rightarrow \frac{3}{4} = 2 + \frac{b_1}{2} - \frac{3}{4}$$

$$\Rightarrow b_1 = 2\left(\frac{3}{4} - 2 + \frac{3}{4}\right) \Rightarrow \boxed{b_1 = -1}$$

$$\boxed{\frac{X^4+2}{X^2(X^2+1)^2} = \frac{2}{X^2} - \frac{1}{X^2+1} - \frac{3}{(X^2+1)^2}}$$