Chapter 3

Polynomials

Definition 36. Let $\mathbb{K}$ be field. We call a polynomial over $\mathbb{K}$ with indeterminate X any expression of the form:

$$P(X) = \sum_{i=0}^{n} a_i X^i = a_0 X^0 + a_1 X^1 + \cdots + a_n X^n \quad \text{where } a_i \in \mathbb{K} \text{ and } a_n \neq 0$$

Definition 37.

1. The elements $a_0, a_1, \cdots, a_n$ are called the coefficients of P.
2. The coefficient a_n is called the leading coefficient. A polynomial is called monic if the leading coefficient is 1.
3. n is called the degree of P and is denoted by $n = degP$.
   if $n = 0$, then $degP = -\infty$
4. The set of all polynomials with coefficients in $\mathbb{K}$ is denoted by $\mathbb{K}[X]$.
5. The set of all polynomials with coefficients in $\mathbb{K}$ of degree less or equal to n is denoted by $\mathbb{K}_n[X]$.

Example 31. $deg(X^3 - 1) = 3 \quad deg(5) = 0 \quad deg(0) = -\infty$

Definition 38. Let $P \in \mathbb{K}[X]$.

$$P(X) = \sum_{i=0}^{n} a_i X^i = a_0 X^0 + a_1 X^1 + \cdots + a_n X^n$$

The polynomial denoted by $P^{(1)}$ and defined as:
$P^{(1)}(X) = \sum_{i=1}^{n} i a_i X^{i-1} = a_1 X^0 + 2a_2 X^1 + \cdots + n a_n X^{n-1}$ is called the derivative of P.

Proposition 1. Let $k \in \mathbb{N}^* \quad degP^{(k)} = degP - k$

Operations on polynomials

Addition

Definition 39. Let $P, Q \in \mathbb{K}[X]$ with: $P(X) = \sum_{i \geq 0} a_i X^i$ and $Q(X) = \sum_{i \geq 0} b_i X^i$

The sum of P and Q denoted $P+Q$ is defined as:

$$(P+Q)(X) = \sum_{i\geq 0} c_i X^i \text{ with } c_i = a_i + b_i$$

Proposition 2.

1. $deg(P+Q) \leq Max(degP, degQ)$
2. $degP \neq degQ \Rightarrow deg(P+Q) = Max(degP, degQ)$

Example 32. $P(X) = -3X^2 + X + 1 \quad Q(X) = 3X^2 - X + 3 \quad (P+Q)(X) = 4$

Multiplication

Definition 40. Let $P, Q \in \mathbb{K}[X]$ with: $P(X) = \sum_{i\geq 0} a_i X^i$ and $Q(X) = \sum_{i\geq 0} b_i X^i$

The product of P and Q denoted PQ is defined as:

$$(PQ)(X) = \sum_{i\geq 0} c_i X^i \text{ with } c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

Proposition 3. deg PQ=deg P+ deg Q

Division

Definition 41. Let $P \in \mathbb{K}[X]$ and , $S \in \mathbb{K}[X] \setminus \{0\}$

$$\boxed{\exists!(Q,R) \in \mathbb{K}[X]^2, P = QS + R \quad \text{with } degR < degS}$$

we call Q the quotient and R the remainder obtained on dividing P by S.
If $R = 0$ we say that S divides P or that P is a multiple of S.

Example 33. $X^3 + X^2 + 3X - 2 = (X+1)(X^2+3) - 5$

$$\begin{array}{rrrr|l}
X^3 & +X^2 & +3X & -2 & X+1 \\ \cline{5-5}
X^3 & +X^2 & & & X^2+3 \\ \hline
 & & 3X & -2 & \\
 & & 3X & +3 & \\ \cline{3-4}
 & & & -5 &
\end{array}$$