# Rings and fields

Let $A$ be a non empty set and $+$, $\times$ two composition laws on $A$ called addition and multiplication respectively.

**Definition 31.** $(A, +, \times)$ is a ring if the following conditions are satisfied:

1. $(A, +)$ is an abelian group, which identity element is denoted by $0$.
2. $\forall x, y, z \in E,\ x \times (y \times z) = (x \times y) \times z$ (we say that $\times$ is associative).
3. $\forall x, y, z \in E,\ x \times (y + z) = (x \times y) + (x \times z)$ (we say that $\times$ is distributive).
   $(y + z) \times x = (y \times x) + (z \times x)$
4. $\times$ has an identity element denoted by 1.

**Remarks** The additive inverse of $x \in A$ is denoted by $-x$.
If the multiplicative inverse of some $x \in A$ exits, it is denoted by $x^{-1}$.

**Example 23.**
$(\mathbb{Z}, +, \times),\ (\mathbb{Q}, +, \times),\ (\mathbb{R}, +, \times),\ (\mathbb{C}, +, \times)$ are rings.
$(\mathbb{N}, +, \times)$ is not a ring.

**Example 24.** $\big(P(E), \Delta, \cap\big)$ is a ring but $\big(P(E), \Delta, \cup\big)$ is not.

**Theorem 10.** Let $x, y \in (A, +, \times)$

1. $x \cdot 0 = 0 \cdot x = 0$
2. $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$
3. $(-x) \cdot (-y) = x \cdot y$

Let $(A, +, \times)$ be a ring and $S$ a subset of $A$.

**Definition 32.** We say that $(S, +, \times)$ is a subring of $A$ if:

1. $(S, +)$ is a subgroup of $(A, +)$.
2. $\forall x, y \in A,\ x \cdot y \in A$
3. $1 \in A$

**Example 25.**
$F = \{z = a + ib : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$.

**Definition 33.** Let $a$ be some nonzero element of a ring $A$.
we say that $a$ is a divisor of zero if there exists $b \neq 0$ such that $ab = 0$.

**Example 26.** Let $(\mathcal{F}, +, \cdot)$ be the ring of real valued functions $\{f : \mathbb{R} \to \mathbb{R}\}$

$$\begin{array}{ccc} f : \mathbb{R} & \to & \mathbb{R} \\ x & \to & f(x) = \begin{cases} x & \text{si } x \geq 0 \\ 0 & \text{si } x < 0 \end{cases} \end{array} \qquad \begin{array}{ccc} g : \mathbb{R} & \to & \mathbb{R} \\ x & \to & g(x) = \begin{cases} 0 & \text{si } x \geq 0 \\ x & \text{si } x < 0 \end{cases} \end{array}$$

Since $fg = 0$, $f$ and $g$ are both divisors of zero.

**Definition 34.** We say that $(A, +, \times)$ is an integral ring if:

1. $1 \neq 0$.
2. $\forall x, y \in A :\ xy = 0 \Rightarrow (x = 0) or (y = 0)$

**Example 27.**
$(\mathbb{Z}, +, \times),\ (\mathbb{Q}, +, \times),\ (\mathbb{R}, +, \times),\ (\mathbb{C}, +, \times)$ are integral rings.
$(\mathcal{F}, +, \times)$ is not an integral ring.

# Fields

**Definition 35.** A commutative ring $(A, +, \times)$ is called a field if every nonzero element of $A$ has a multiplicative inverse.

**Example 28.**

$(\mathbb{Q}, +, \times),\ (\mathbb{R}, +, \times),\ (\mathbb{C}, +, \times)$ are fields.

$(\mathbb{Z}, +, \times)$ is not a field.

**Example 29.**

$F = \{z = a + ib : a, b \in \mathbb{Z}\}$ is not a field.

**Example 30.**

$F = \{z = a + ib : a, b \in \mathbb{Q}\}$ is a field.