Mappings

Definition 19. Let A and B be nonempty sets. A mapping (or map) f from A to B denoted $f : A \to B$ is a rule that associates with each element x of the set A exactly one element y of the set B.

If $a \in A$, then the element of B that is associated with a is denoted by $f(a)$ and is called the image of a under f. If $f(a) = b$, with $b \in B$, then a is called a preimage of b under f.

The set A is called the domain of the map f , and we write $A = dom(f)$. The set B is called the codomain of the map f , and we write $B = codom(f)$.

Example 10. $f : \begin{array}{ccc} \mathbb{R}^+ & \to & \mathbb{R} \\ x & \mapsto & \sqrt{x} \end{array}$ is a mapping.

$g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \sqrt{x} \end{array}$ is not a mapping.

$Id_E : \begin{array}{ccc} E & \to & E \\ x & \mapsto & x \end{array}$ is called the identity map.

Image, pre-image

Definition 20. Let $f : A \to B$ and $G \subseteq A$. The image of G under f is the set:

$$f[G] = \{f(x) \mid x \in G\}$$

$f[A]$ is said to be the range of f and is denoted by $Ran(f) = f[A]$

Definition 21. Let $f : A \to B$ and $b \in B$. The pre-image or the inverse image of b under f is the set:

$$f^{-1}(b) = \{x \in A \mid f(x) = b\}$$

Definition 22. Let $f : A \to B$ and $H \subseteq B$. The pre-image or the inverse image of H under f is the set:

$$f^{-1}[H] = \{x \in A \mid f(x) \in H\}$$

Example 11. Let $f : \begin{array}{ccc} \mathbb{R}^+ & \to & \mathbb{R} \\ x & \mapsto & \sqrt{x} \end{array}$ be a mapping.

Calculate $f[G]$ and $f^-1[H]$ where $G = [1, 2[$ and $H =]3, 5]$
$1 \leq x < 2 \Rightarrow 1 \leq \sqrt{x} < \sqrt{2} \Rightarrow 1 \leq f(x) < \sqrt{2} \Rightarrow f[G] = [1, \sqrt{2}[$
$3 < f(x) \leq 5 \Rightarrow 3 < \sqrt{x} \leq 5 \Rightarrow 3^2 < x \leq 5^2 \Rightarrow f^-1[H] =]9, 25]$

Composition of maps

Definition 23. Let $f : A \to B$ and $g : B \to C$. Then the composition of g with f is the map $g \circ f : A \to C$ defined by

$$(g \circ f)(x) = g(f(x))$$

Example 12. $f : \begin{array}{ccc} [-1, +\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x+1} \end{array}$ $\quad g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}_+ \\ x & \mapsto & x^2 \end{array}$ $\quad g \circ f : \begin{array}{ccc} [-1, +\infty[& \to & \mathbb{R}_+ \\ x & \mapsto & x+1 \end{array}$

Remark In general, $g \circ f \neq f \circ g$.

Injection, Surjection, inverse map

Definition 24. A map $f : A \to B$ is said to be injective or simply f is an injection if

$$\forall x_1, x_2 \in A, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

In practice, we often use the contrapositive:

$$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Definition 25. A map $f : A \to B$ is said to be surjective or simply f is a surjection provided that

$$f[A] = B \Leftrightarrow \forall y \in B, \exists x \in A \mid y = f(x)$$

Example 13. $f : \begin{array}{ccc} [-1, +\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x+1} \end{array} \qquad g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}_+ \\ x & \mapsto & x^2 \end{array}$

f is injective but g is not.
g is surjective but f is not.

Definition 26. A map $f : A \to B$ that is both injective and surjective is said to be bijective. This means that

$$\forall y \in B, \exists!\, x \in A \mid y = f(x)$$

Theorem 5. Let $f : A \to B$ be a bijection. Then there exists a unique map called the inverse of f and denoted $f^{-1} : B \to A$ such that:

1. $\forall y \in B :\ (f \circ f^{-1})(y) = y$
2. $\forall x \in A :\ (f^{-1} \circ f)(x) = x$

Example 14. $h : \begin{array}{ccc} [-1, +\infty[& \to & \mathbb{R}_+ \\ x & \mapsto & \sqrt{x+1} \end{array}$ is bijective and its inverse is given by:

$$h^{-1} : \begin{array}{ccc} \mathbb{R}_+ & \to & [-1, +\infty[\\ y & \mapsto & y^2 - 1 \end{array}$$

Theorem 6. Let $f : A \to B$ and $g : B \to C$. Then

1. if f and g are both injections, then $g \circ f$ is an injection.
2. if f and g are both surjections, then $g \circ f$ is a surjection.
3. if f and g are both bijections, then $g \circ f$ is a bijection. and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Theorem 7. Let $f : A \to B$ and $g : B \to C$. Then

1. $(g \circ f)$ injective $\Rightarrow f$ injective
2. $(g \circ f)$ surjective $\Rightarrow g$ surjective

Example 15. $f : \begin{array}{ccc} [-1, +\infty[& \to & \mathbb{R} \\ x & \mapsto & \sqrt{x+1} \end{array}$ $\quad g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}_+ \\ x & \mapsto & x^2 \end{array}$ $\quad g \circ f : \begin{array}{ccc} [-1, +\infty[& \to & \mathbb{R}_+ \\ x & \mapsto & x+1 \end{array}$

- $g \circ f$ and f are injective but g is not.
- $g \circ f$ and g are surjective but f is not.