Nonlinear Systems

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Chapter 1

Introduction to theory: basic definitions

A general nonlinear dynamic system may be modeled by a finite number of coupled first-order ordinary differential equations.

$$\dot{x} = f(t, x, u) \tag{1}$$

where $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$: state vector, **n**: system order,
$u = [u_1, \cdots, u_m]^T \in \mathbb{R}^m$: input vector (control, disturbance).

- **f(.)** is a vector field in $\mathbb{R}^n$: a function associating a vector to n-dim point **x**.
- **Initial condition**: $x(t_0) = (x_1(0), x_2(t_0), \cdots, x_n(t_0))$
- Eq. (1) is called **state equation**.
- **Solution** is in the form $x(t_0, t)$ that defines a family of time trajectories in the state space (also referred as phase space).Imposing the initial condition $x(t_0)$ determines one unique trajectory.
- Another equation named **output equation**:

$$y = h(t, x, u) \tag{2}$$

where $y = [y_1, \cdots, y_p]^T \in \mathbb{R}^p$: output vector.

Eqs. (1) and (2) together are called **state-space model**.

Introduction to theory: basic definitions

Analysis : unforced state equations

$\dot{x} = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$ time-invariant (autonomous)

$\dot{x} = f(t, x)$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ time-varying (Non-autonomous)

Control design:

$\dot{x} = f(x, u)$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ autonomous with inputs

$\dot{x} = f(t, x, u)$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ Non-autonomous with inputs

$\dot{x} = f(x) + g(x)u$, $f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^m \to \mathbb{R}^n$ affine in u

Linear systems: if f and h are linear functions of x and u

$$\dot{x} = f(x) \quad \to \quad \dot{x} = Ax \qquad \text{autonomous}$$

$$\dot{x} = f(x, u) \quad \to \quad \dot{x} = Ax + Bu$$

$$\dot{x} = f(t, x, u) \quad \to \quad \dot{x} = A(t)x + B(t)u$$

Introduction to theory: basic definitions

For linear systems

The Linear Systems satisfy the superposition principals:

1. Homogeneity: $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in \mathbb{R}$.
2. Additivity: $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}^n$.

Unique equilibrium point.

Stability is independent of initial conditions $x(t) = x(0)e^{At}$

Introduction to theory: basic definitions

For nonlinear systems

Non superposition principle $\Longrightarrow$ More complex behavior

Example : Under-water vehicle $\dot{v} + |v|v = u$

Settles faster in response to positive step.

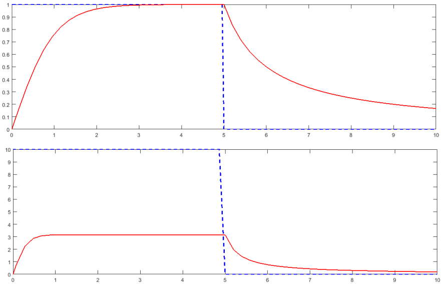
Scaling input does not result into the same scaling in output.

$u = 1$

$$\begin{aligned} \Rightarrow 0 + |v_s|v_s &= 1 \\ \Rightarrow v_s &= 1 \end{aligned}$$

$u = 10$

$$\begin{aligned} \Rightarrow 0 + |v_s|v_s &= 10 \\ \Rightarrow v_s &= \sqrt{10} \end{aligned}$$



Nonlinear System Examples

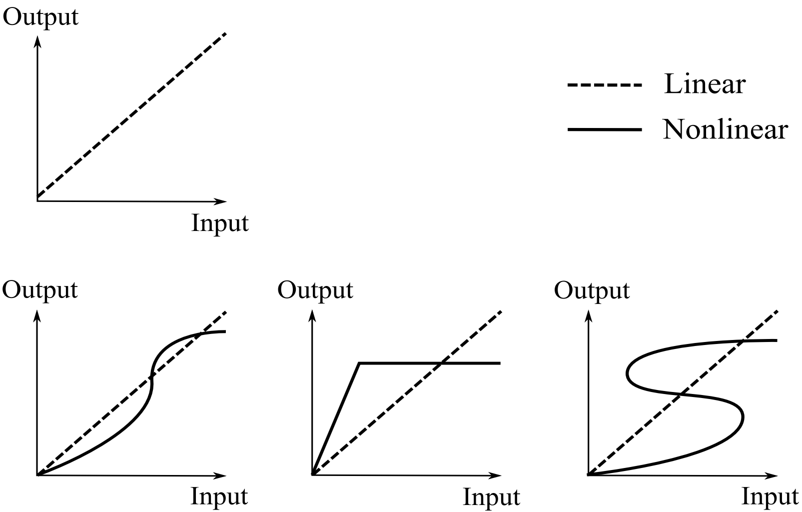


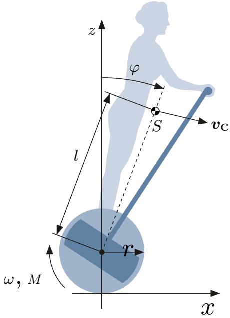
Figure: A nonlinear system is a system in which the change of the output is not proportional to the change of the input.

Nonlinear System Examples

Systems with essential nonlinearities in the model

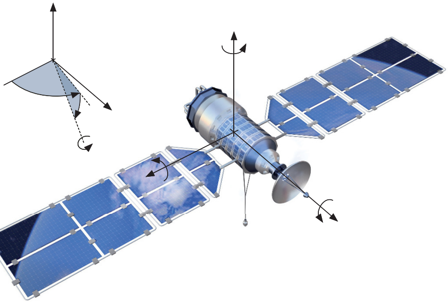
Self-balancing vehicle

$$\ddot{x} = \frac{c}{a}\;\; \dot{\varphi}^2 \sin(\varphi) - \ddot{\varphi}\cos(\varphi)\big) + \frac{M}{ar},$$
$$\ddot{\varphi} = \frac{c}{b}\left(g\sin(\varphi) - \ddot{x}\cos(\varphi)\right) - \frac{M}{b},$$



Euler's rotation equations

$$J_x\dot{\omega}_x = -(J_z - J_y)\omega_y\omega_z + M_x$$
$$J_y\dot{\omega}_y = -(J_x - J_z)\omega_x\omega_z + M_y$$
$$J_z\dot{\omega}_z = -(J_y - J_x)\omega_x\omega_y + M_z$$

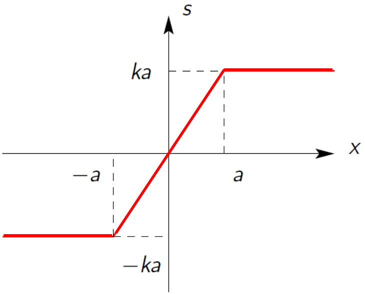


Nonlinear System Examples

Systems with saturation

$$\begin{aligned} \dot{x} &= Ax + Bsat(u) \\ u &= PID(x) \end{aligned}$$

$$sat(u) = \begin{cases} u & \text{If} \quad |u| \leq 1 \\ sgn(u) & \text{If} \quad |u| \geq 1 \end{cases}$$

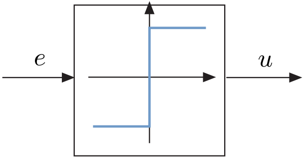


The output is proportional to input for a limited range.

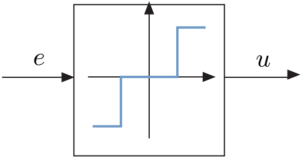
Output becomes constant if input is outside this range.

Common Nonlinearities

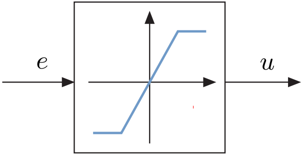
memoryless



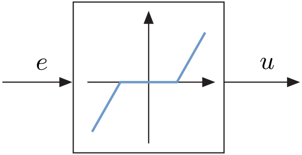
Two-position element



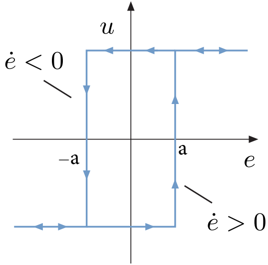
Three-position element



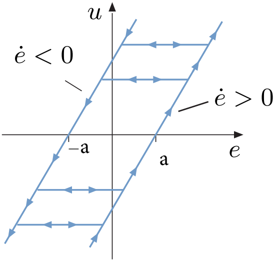
(Saturation characteristic)



Dead zone



Hysteresis characteristic curve



Backlash characteristic curve

Equilibrium points

An **equilibrium point** represents a stationary condition of a dynamical system. The state $x_e \in \mathbb{R}^n$ is a fixed point for $\dot{x} = f(x)$ if $f(x_e) = 0$, $\forall t \geq 0$.

If a dynamical system has an initial condition $x(0)$ at an equilibrium point x_e, then it will stay at x_e forever, i.e. $x(t) = x_e$, $\forall t \geq 0$.

Example 1: $\dot{x} = x(x-2)^2$

$$f(\bar{x}) = 0 \Rightarrow x(x-2)^2 = 0 \Rightarrow \begin{cases} x_e & = & 0 \\ x_e & = & 2 \end{cases}$$

This system has two isolated equilibrium points at 0 and 2.

Example 2:

$$\dot{x} = \sin(x), \quad x(0) = x_0 \in \mathbb{R} \tag{3}$$

$x(t) \in \mathbb{R}$ 1st order system (scalar state)

$$f(\bar{x}) = \sin(\bar{x}) = 0 \Longrightarrow \bar{x} = k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots$$

This system has infinite many equilibrium points.

Equilibrium points

Non-isolated equilibria

For a linear system $\dot{x} = Ax, \qquad A \in \mathbb{R}^{n\times n}$.

If A is nonsingular (det(A)$\neq 0$), then $x^* = 0$ is the unique equilibrium.

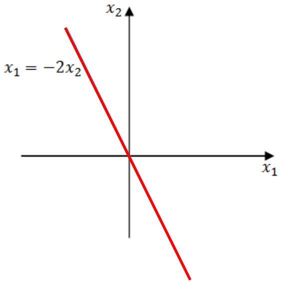
If A is singular (det(A)$= 0$), then Null space defines a continuum of equilibria.

Example:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

$$\dot{x} = 0 \Rightarrow Ax = 0 \Rightarrow \begin{cases} -2x_1 - 4x_2 &= 0 \\ -x_1 - 2x_2 &= 0 \end{cases}$$

$$\Rightarrow x_1 = -2x_2$$



Linear approximations around an equilibrium

A linear approximation of a system around an equilibrium point can be used to study the behavior of a nonlinear system around the equilibrium point.

Local stability properties of x_e can be determined by linearizing the vector field $f(x)$ at x_e:

$$f(x_e + \tilde{x}) = \underbrace{f(x_e)}_{=0} + \underbrace{\left.\frac{\delta f}{\delta x}\right|_{x=\bar{x}}}_{\triangleq A} \tilde{x} + \text{higher order terms in } (\mathrm{x} - \mathrm{x_e}) \quad (4)$$

Thus, the linearized model is :

$$\dot{\tilde{x}} = A\tilde{x} \quad (5)$$

The solution of the linearized form (5) has the form : $\tilde{x} = e^{At}x_0$

- If Re$(\lambda_i(A)) < 0$, then x_e is locally asymptotically stable.
- If Re$(\lambda_i(A)) > 0$ for some eigenvalue λ_i of A, then $\bar{x}$ is unstable.

Linear approximations around an equilibrium

Example 1: Linearization of $\dot{x} = \sin(x)$

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \cos(x)|_{\bar{x}} = \begin{cases} \cos(2K\pi) & = & 1, & n \text{ even} \\ \sin((2K+1)\pi) & = & -1, & n \text{ odd} \end{cases}$$

$\bar{x} = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$.

Example 2: (2nd order system)

$$\dot{x} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + x_1^2 x_2 - x_2 \end{bmatrix}$$

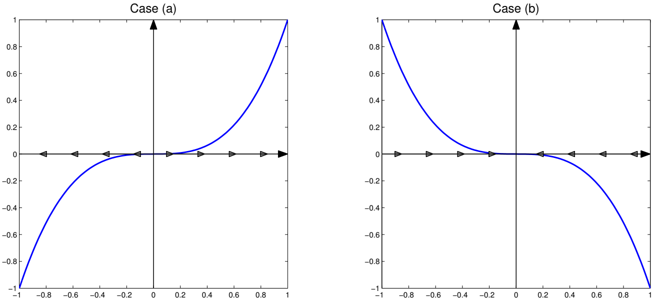
$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x_1 x_2 & x_1^2 - 1 \end{bmatrix}$$

Linearization around $x_e(0,0)$: $\left.\frac{\delta f}{\delta x}\right|_{x_e} (x - x_e) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Caveats:

1. Only local properties can be determined from the linearization.
2. If $Re(\lambda_i) \leq 0$ with some e-values having $Re(\lambda_i) = 0$, then linearization is inconclusive as a stability test. Higher order terms determine stability.

Example: $(a) : \dot{x} = x^3, \quad (b) : \dot{x} = -x^3.$
In both cases, linearized systems around $\bar{x} = 0$ are the same: $\dot{x} = 0 \Rightarrow x(t) = x_0$, but NL systems have different behaviors.



Essentially Nonlinear Phenomena

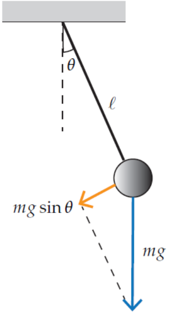
1. Multiple Isolated equilibrium

Linear systems have only one equilibrium point at origin or a number of non isolated equilibrium points.

Example: Pendulum (two isolated equilibia)

$$lm\ddot{\theta} = -kl\dot{\theta} - mg\sin(\theta) \qquad (6)$$

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space : $S^1 \times \mathbb{R}$



$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 \end{aligned} \qquad (7)$$

Equilibria: Two isolated E.P (0,0) and (π, 0).

Linearization :

$$\frac{\delta f}{\delta x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{l} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{l} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{l} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases} \qquad (8)$$

Stability may depend upon initial conditions

Example : The logistic equation (population dynamics in isolation)

$$\dot{x} = f(x) = \underbrace{rx(k - x)}_{\text{growth rate}} x, \qquad r > 0, K > 0 \tag{9}$$

$x \in \mathbb{R}_+$ is the population size and K is the carrying capacity.
Stability can be determined from the sign of $f(x)$ around the equilibrium.

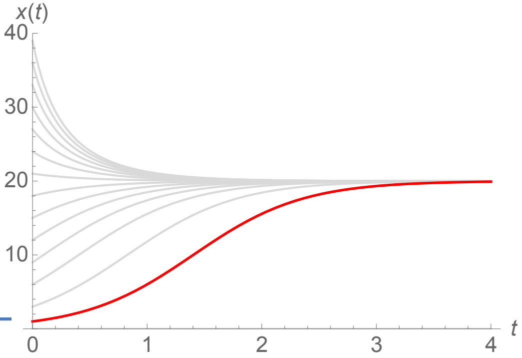
Equilibrium points are : $(\dot{x}_e = 0)$

$$\Rightarrow rx_e(K - x_e) = 0 \Rightarrow \begin{cases} x_e = 0 \\ x_e = K \end{cases}$$

Linearization :

$$\left.\frac{\partial f}{\partial x}\right|_{x_e} = (rK - 2rx)|_{x_e}$$

$$\begin{cases} \bar{x} = 0 \rightarrow Kr \rightarrow \text{unstable.} \\ \bar{x} = K \rightarrow -Kr \rightarrow \text{stable} \end{cases} \tag{10}$$



2. Finite Escape Time

In linear case :

$$\dot{x} = \lambda x \quad \xrightarrow{Solution} \quad x(t) = \exp^{\lambda t} x(0).$$

If $\lambda > 0 \Longrightarrow \lim_{t\to\infty} |x(t)| = +\infty.$ Only as $t \to \infty$, $|x(t)| \to \infty$.

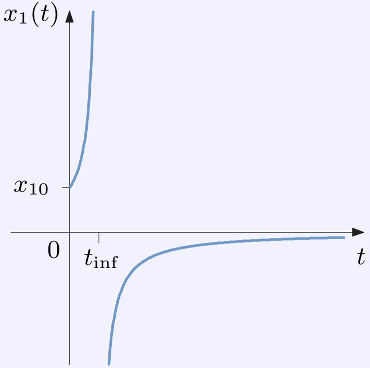
In nonlinear case :

Example : $\dot{x} = x^2$, $x(0) = x_0$, $x \in \mathbb{R}$

$$\frac{dx}{dt} = x^2 \quad \Rightarrow \quad \int_{x_0}^{x(t)} \frac{dx}{x^2} = \int_0^t dt$$
$$\Rightarrow \quad -\frac{1}{x(t)} + \frac{1}{x_0} = t - 0$$

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

$x_0 > 0 \Rightarrow t \to \frac{1}{x_0} \Rightarrow x(t) \to \infty$



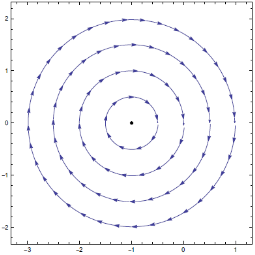
3. Limit cycles

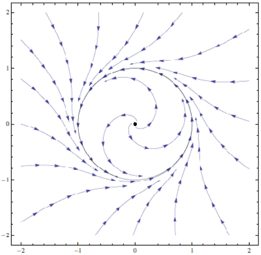
Linear oscillators exhibit a continuum of periodic orbits (closed orbit) :

$$x(t+T) = x(t),\ \forall t > 0, \text{for somme } T > 0$$

Every circle is periodic orbit for $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}, \quad (\lambda_{1,2} = \pm j\beta)$$



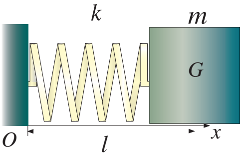


Every Periodic Orbit is a Cycle but not a Limit Cycle.
In contrast, a limit cycle is an isolated closed trajectory (closed orbit) and can occur only in nonlinear systems.

3. Limit cycles

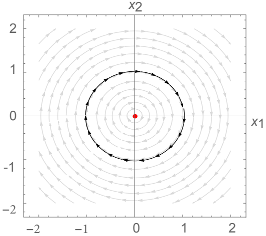
Example: Harmonic oscillator

$$\underbrace{m\ddot{x}}_{\text{inertial term}} + \underbrace{kx}_{\text{stiffness term}} = 0 \qquad (11)$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\omega_0 = \sqrt{\frac{k}{m}}.\ (\lambda_{1,2} = \pm j\omega_0).$$



Amplitude of oscillations depends on initial conditions.

Can be destroyed by small modelling imperfections.

> The harmonic oscillator has closed orbits but no limit cycles. Limit cycles cannot be generated by LTI systems.
>
> The linear oscillator is not structurally stable. A stable oscillators must be produced by nonlinear systems.

3. Limit cycles

Example : Van der Pol oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

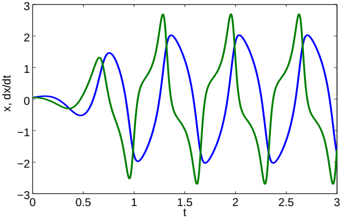
If $\mu = 0 \Rightarrow \ddot{x} + x = 0 \leftarrow$ simple harmonic oscillator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1 \end{cases} \qquad (12)$$

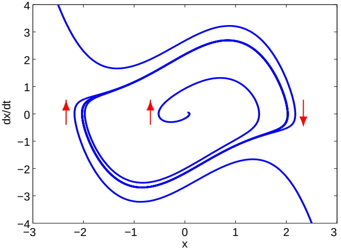
Equilibrium point :$\dot{x} = 0 \Rightarrow \bar{x} = [0\ 0]^T$.
Linarization around $\bar{x} = (0, 0)$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

Positive sign of μ tells us that e.p is unstable.



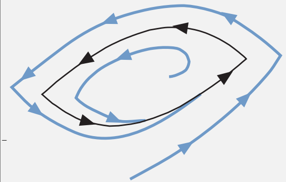
Response with $x(0) = 0.05$, $x^{\cdot}(0) = 0.05$



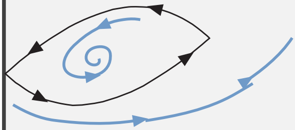
State trajectories $(x(t), \dot{x}(t))$

3. Limit cycles: Examples of limit cycles

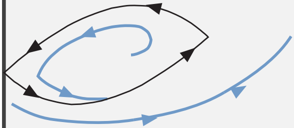
Stable Limite Cycle



Unstable Limite Cycle



SemiStable Limite Cycle



4. Chaos

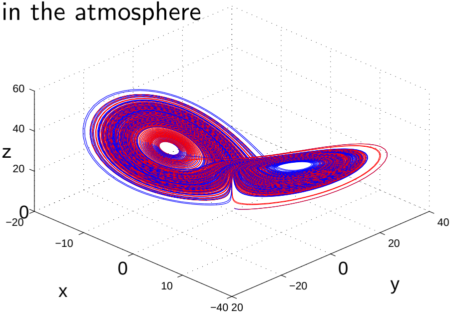
Chaos

Irregular oscillations, never exactly repeating.
Behavior of nonlinear systems may be extremely sensive to small changes in initial conditions/input/parameters.

Example : Lorenz system (attractor) derrived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere

The lorenz system is a 3rd order system (3 states x,y,z).

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \qquad (13)$$



4. Chaos

For continuous-time, time invariant systems, $n \geq 3$ state variable required for chaos.

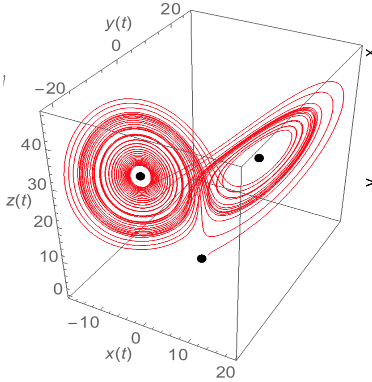
No simple characterization of asymptotic behavior.

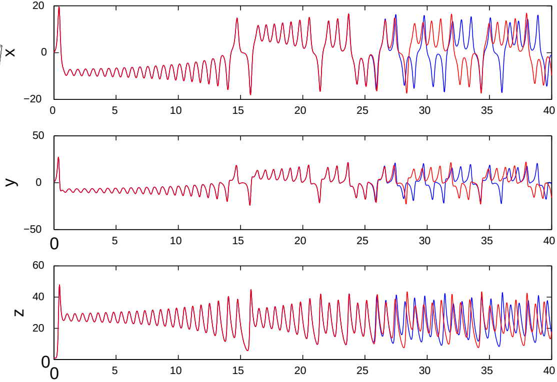
Huge sensitivity to initial conditions.

Choatic behavior with $\sigma = 10$, b=8/3, r=28

blue: $(x, y, z) = (0, 1, 1.05)$

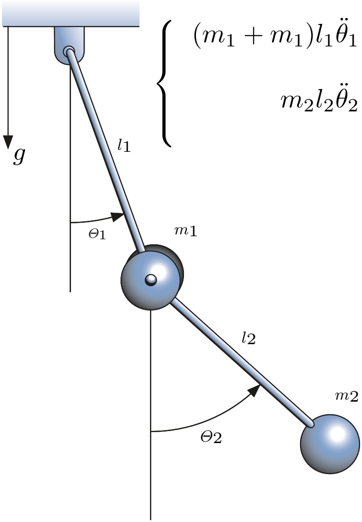
red: $(x, y, z) = (0, 1, 1.050001)$



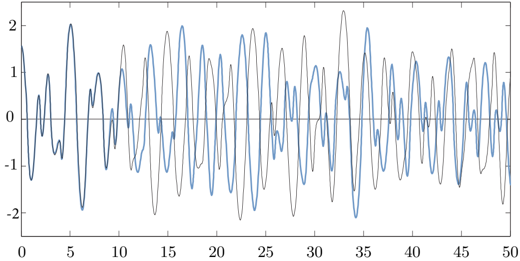


4. Chaos

Example: The double pendulum (System is implicit for $l_1 \neq l_2$)



$$\begin{cases} (m_1 + m_1) l_1 \ddot{\theta}_1 & + \quad m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}^2 \sin(\theta_1 - \theta_2) \\ & + \quad g(m_1 + m_2) \sin\theta_1 = 0 \\ m_2 l_2 \ddot{\theta}_2 & + \quad m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \\ & + \quad m_2 g \sin\theta_2 = 0 \end{cases}$$



Time t in s

5. Bifurcation: Fold bifurcation

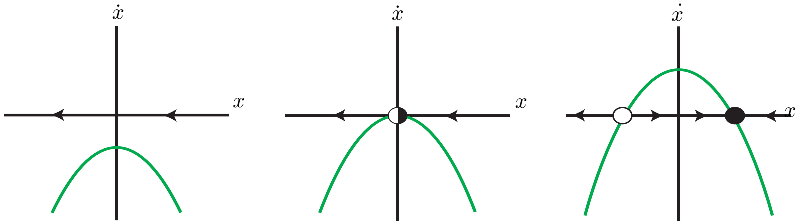
A **bifurcation** is an abrupt change in qualitative behavior as a parameter is varied. Examples : creation (or death) of equilibrium points (or limit cycles) and/or change of their stability properties.

Fold bifurcation: 1st order system

Example : $\dot{x} = \mu - x^2$,

Equilibrium points :

$$\bar{x} = \begin{cases} \pm\sqrt{\mu} & \mu > 0 \quad \text{one stable equilibrium and one unstable equilibrium} \\ 0 & \mu = 0 \quad \text{single equilibrium (called a saddle)} \\ \text{none} & \mu < 0 \quad \text{no equilibria} \end{cases}$$



5. Bifurcation: Fold bifurcation

$\mu_c = 0$ is the critical value of parameter μ which represents boundary between "no equilibrium points" and the presence of equilibrium points.

Creation/destruction of fixed points is called **saddle node bifurcation**

Linearization

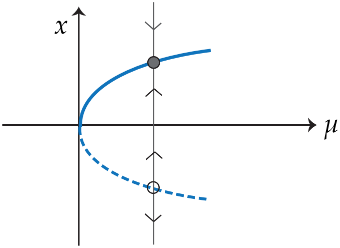
$$\left.\frac{\delta f}{\delta x}\right|_{\bar{x}} = 2\bar{x} = \begin{cases} 2\sqrt{\mu} & \text{unstable} \\ -2\sqrt{\mu} & \text{stable} \end{cases}$$

$$\mu > 0.$$

Note:

$A_c = \left.\frac{\delta f}{\delta x}\right|_{\bar{x}_c = \bar{x}(\mu_c)} = 0 \rightarrow$ linearization disappears, no information about stability of the system.

bifurcation diagram



5. Bifurcation: Transcritical bifurcation

- **Transcritical bifurcation:**

Example : $\dot{x} = \mu x - x^2, \quad x(t) \in \mathbb{R}$

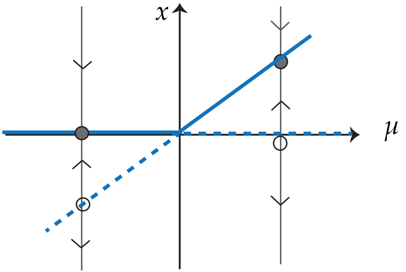
Equilibrium points : $\bar{x} = 0. \quad \bar{x} = \mu.$

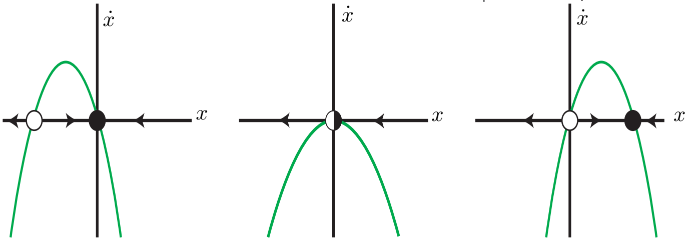
Linearization :

$$\frac{\delta f}{\delta x} = \mu - 2\bar{x} = \begin{cases} \mu & \text{if} \quad \bar{x} = 0 \\ -\mu & \text{if} \quad \bar{x} = \mu \end{cases}$$

$\mu < 0$: $\bar{x} = 0$ is stable, $\bar{x} = \mu$ is unstable.
$\mu > 0$: $\bar{x} = 0$ is unstable, $\bar{x} = \mu$ is stable.

bifurcation diagram





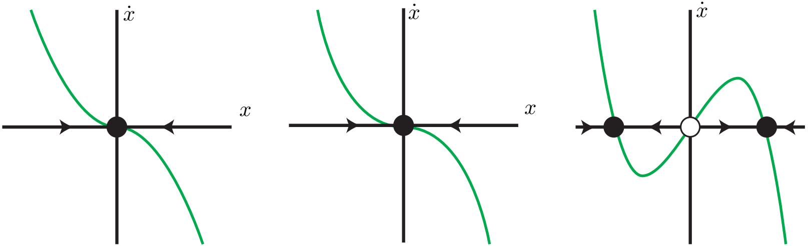
5. Bifurcation : Pichfork bifurcation

Pitchfork Bifurcation- 2 types : supercritical Pitchfork and subcritical pitchfork

Example 1:

$$\dot{x} \quad = \quad \mu x - x^3 \quad \text{supercritical}$$

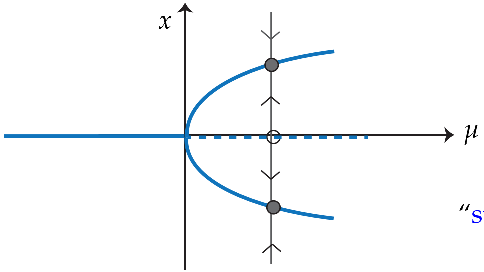
Equilibrium points : $f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{\mu}, & \mu > 0 \end{cases}$



2 equilibrium points emerge when we increase μ.

5. Bifurcation : Pitchfork bifurcation

bifurcation diagram



"supercritical pitchfork"

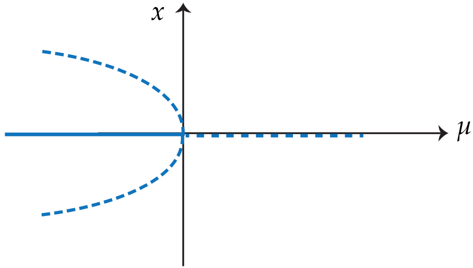
Example 2: $\dot{x} = \mu x + x^3$, subcritical pitchfork.
Equilibrium points :

$$f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{-\mu}, & \mu < 0 \end{cases}$$

5.Bifurcation: Pitchfork bifurcation

Linearization :

$$\left.\frac{\delta f}{\delta x}\right|_{\bar{x}=0} = \mu$$
$$\left.\frac{\delta f}{\delta x}\right|_{\bar{x}=\pm\sqrt{-\mu}} = -2\mu$$



“subcritical pitchfork"

Chapter 2: Second Order Systems

Concept of Phase Plane

A second-order autonomous system is represented by two scalar differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{1}$$

Let $x(t) = (x_1(t), x_2(t))$ be the solution of (1) that starts at a certain initial state $x_0 = (x_{10}, x_{20})$.

f(.) is called a **vector field**

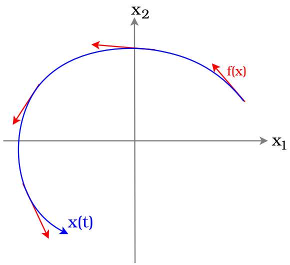
The set of points $\{(t, x_1(t), x_2(t)); t \in \mathbb{R}\}$ with (x_1, x_2) a solution of (1) (and $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$ for some t_0) is called the **trajectory** or **solution curve** (through (x_{10}, x_{20})).

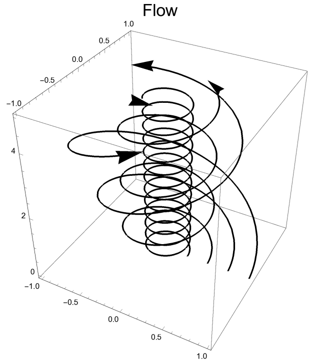
The set of points $\{(x_1(t), x_2(t)); t \in \mathbb{R}\}$ with (x_1, x_2) a solution of (1) (and $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$ for some t_0) is called the $fOrbit$ or **Phase Curve** (through (x_{10}, x_{20})).

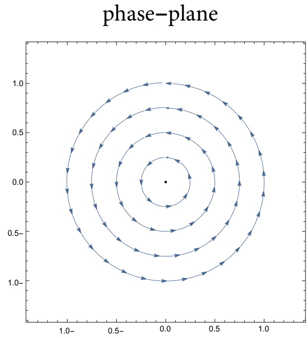
Concept of Phase Plane

An orbit that forms a closed curve is called a **closed orbit**.

The family of all trajectories of a dynamical system is called **the phase portrait.**







Vector fields and Orbits

Example: Uncoupled system

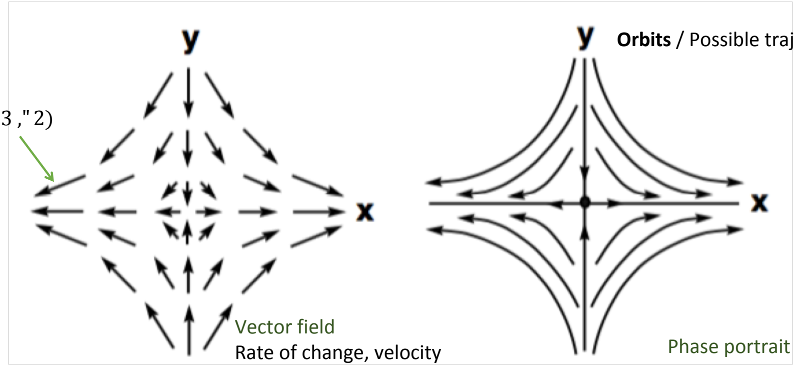
$$\begin{cases} \dot{x} &= 2x &= f_x(x,y) \\ \dot{y} &= -3y &= f_y(x,y) \end{cases}$$

f is a **vector field** in $\mathbb{R}^2$.

Solution : $(x_0 e^{2t}, y_0 e^{-3t})$

. = (−3 ," 2)

Direction and *speed* of solution for any (x, y)

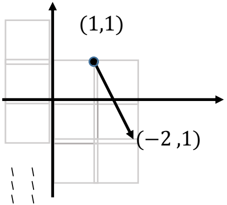


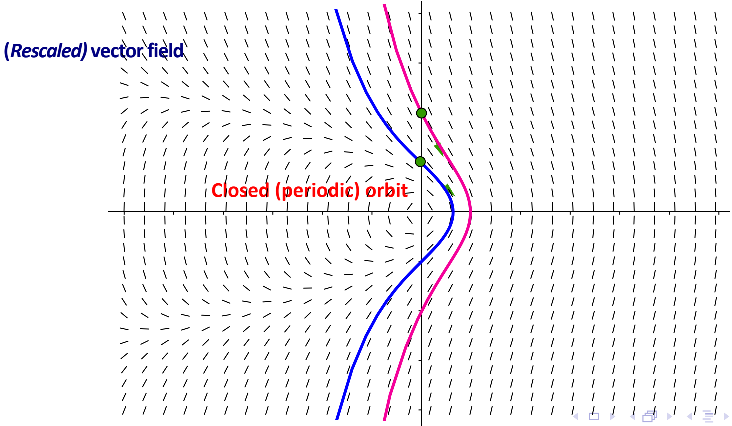
Orbits / Possible trajectories

Vector fields and Orbits

Example

$$\begin{cases} \dot{x} &= y &= f_x(x,y) \\ \dot{y} &= -x - y^2 &= f_y(x,y) \end{cases} \qquad \begin{matrix} (x,y) & \rightarrow & f(x, -x-y^2) \\ (1,1) & \rightarrow & (1,-2) \end{matrix}$$

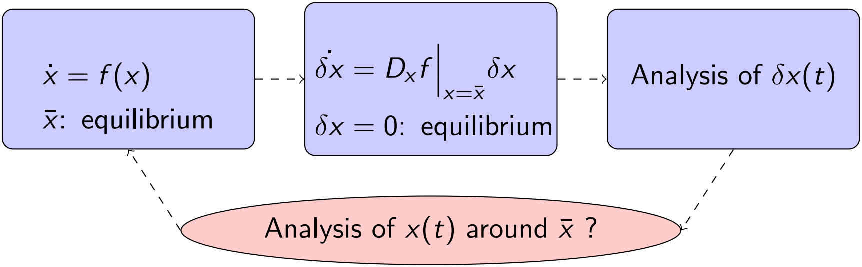




Phase plan analysis

Problem

When $x(t) \in \mathbb{R}^2$, study state trajectories around an equilibrium state



Behavior of Linear second Order Systems

Consider the following linear system

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

$(a, b, c, d) \in \mathbb{R}$. Change of coordinates: $z(t) = T^{-1}x(t)$, $T \in \mathbb{R}^{2\times 2}$ invertible.

$$\dot{z}(t) = T^{-1}\dot{x}(t) = T^{-1}Ax(t) = T^{-1}ATz(t) = Jz(t)$$

The system $\dot{z} = Jz$ is equivalent to the system $\dot{x} = Ax$.

Remark

A and $J = T^{-1}AT$ are similar $\Longrightarrow$ they have the same eigenvalues

One can always choose T such that J is in real Jordan form

the new coordinates are called normal

Behavior of Linear second Order Systems

There are three possible Jordan forms for A:

Different real eigenvalues.

Equal real eigenvalues.

Complex conjugate.

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} \alpha & \beta \\ -\beta & \lambda \end{bmatrix}$$

where k=0 or 1.

In addition, we need to consider the case where at least one of the eigenvalues is zero.

Behavior of Linear second Order Systems

Case 1: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 \in \mathbb{R}$, and independent eigenvectors

In this case

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \Longrightarrow T = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the real eigenvectors of A associated with λ_1 and λ_2, respectively.

The change of coordinate $z = T^{-1}x$, transforms the system into two decoupled first-order differential equations, i.e.,

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_1 z_2$$

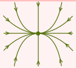
with solution

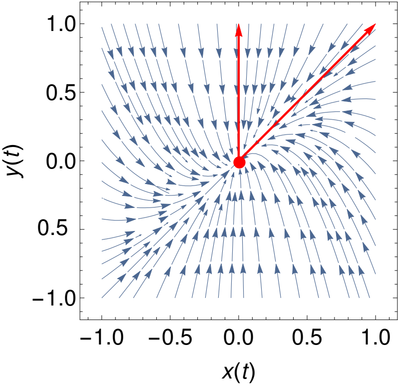
$$z_1(t) = z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t} \Longrightarrow z_2(t) = \frac{z_{20}}{(z_{10})^{\lambda_2/\lambda_1}} z_1^{\lambda_2/\lambda_1}$$

Behavior of Linear second Order Systems

Case 1a: $\lambda_1 < 0$ and $\lambda_2 < 0$

The origin is called **stable node**





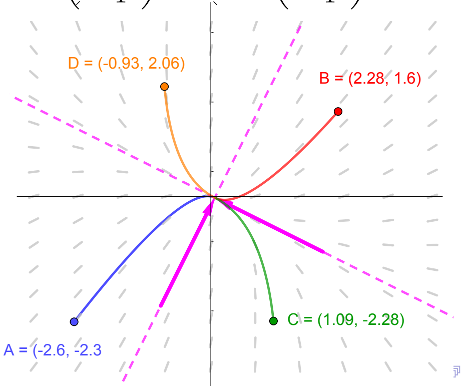
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned}\dot{x} &= -6x - 2y \\ \dot{y} &= -2x - 9y\end{aligned}$$

eigenvalues : $\lambda_1 = -10; \lambda_2 = -5 \Longrightarrow$ *stable node*

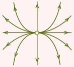
eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix},$

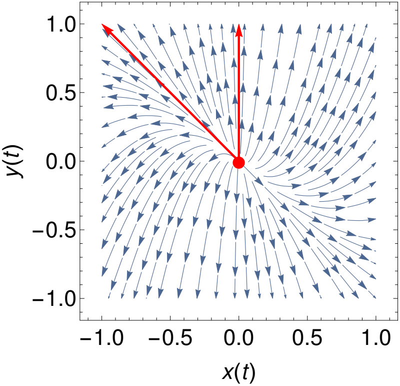


Behavior of Linear second Order Systems

Case 1b: $\lambda_1 > 0$ and $\lambda_2 > 0$

The origin is called **Unstable Node**





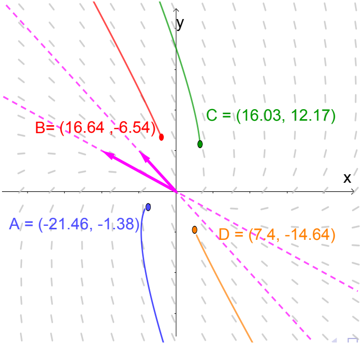
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= x - 2y \\ \dot{y} &= x + 4y \end{aligned}$$

eigenvalues : $\lambda_1 = 2;\ \lambda_2 = 3 \Longrightarrow$ *unstable node*

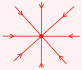
eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$

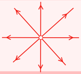


Behavior of Linear second Order Systems

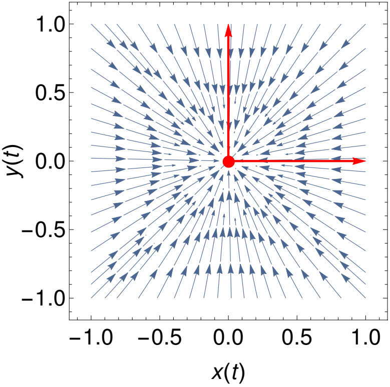
Case 1d: $\lambda_1 = \lambda_2$

The origin is called **stable/unstable degenerate node**

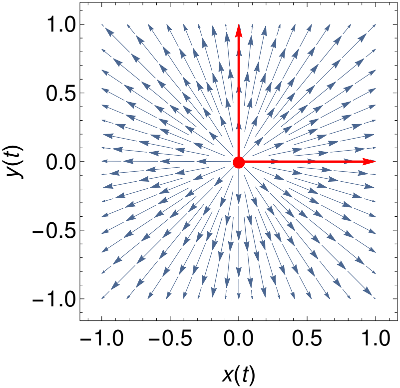




Stable degenerate node: $\lambda_1 = \lambda_2 < 0$



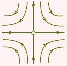
Unstable degenerate node: $\lambda_1 = \lambda_2 > 0$

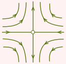


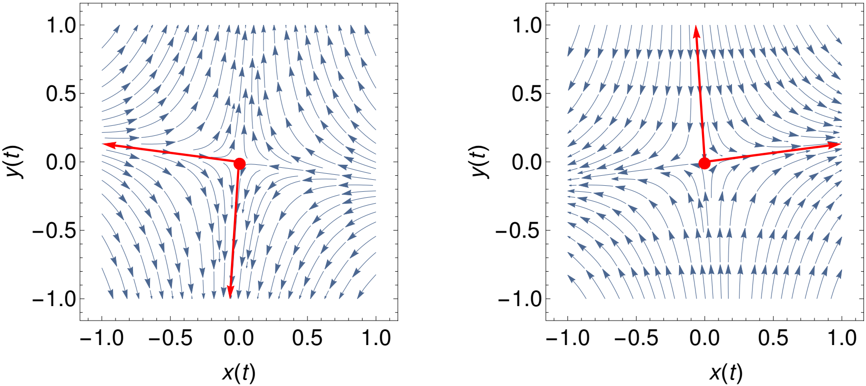
Behavior of Linear second Order Systems

Case 1b: $\lambda_1 < 0 < \lambda_2$

The origin is called **saddle**







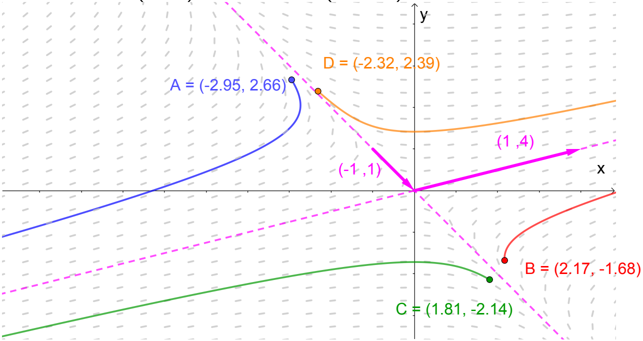
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= 3x + 4y \\ \dot{y} &= x \end{aligned}$$

eigenvalues : $\lambda_1 = 4; \lambda_2 = -1 \Longrightarrow$ *Saddle*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$

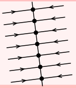


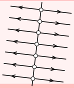
$$\dot{z}_1 = \lambda_1 z_1 \rightarrow \dot{z}_1(t) = z_1(0)e^{\lambda_1 t}$$
$$\dot{z}_2 = \lambda_2 z_2 \rightarrow \dot{z}_2(t) = z_2(0)e^{\lambda_2 t}$$

Case 1e: degenerate saddle

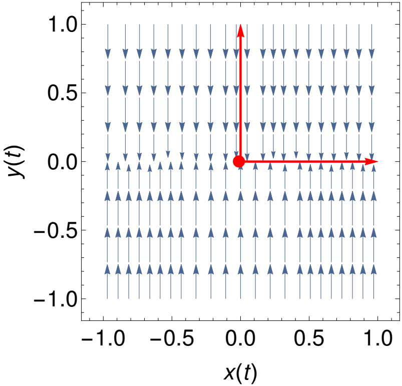
$\lambda_1 < \lambda_2 = 0 \rightarrow$ all states on the z_2 axis are equilibrium states.

$0 = \lambda_1 < \lambda_2 \rightarrow$ all states on the z_1 axis are equilibrium states.

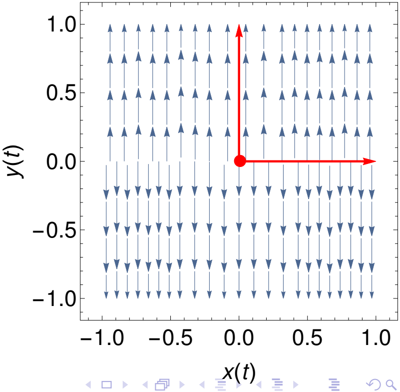




$\lambda_1 < \lambda_2 = 0$



$0 = \lambda_1 < \lambda_2$



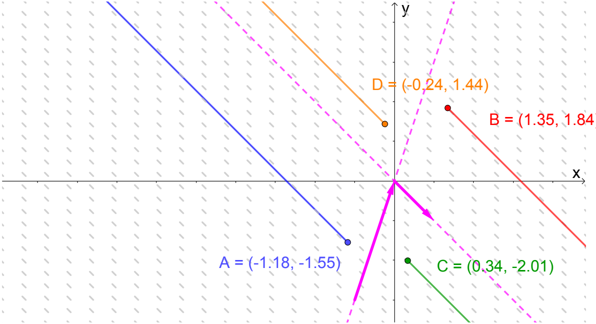
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= 3x - y \\ \dot{y} &= -3x + y \end{aligned}$$

eigenvalues : $\lambda_1 = 4; \lambda_2 = 0 \Longrightarrow$ *degenerate Source*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 0.3333 \\ 1 \end{pmatrix}$



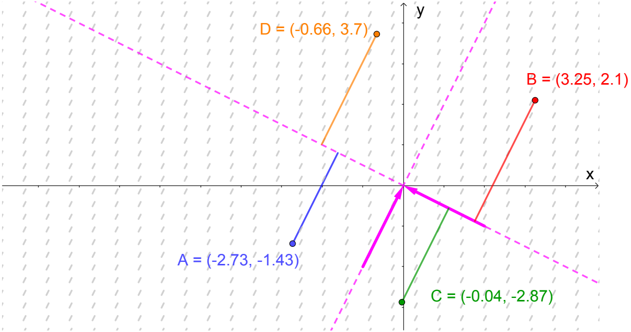
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= x - 2y \\ \dot{y} &= -2x - 4y \end{aligned}$$

eigenvalues : $\lambda_1 = -5; \lambda_2 = 0 \Longrightarrow$ *Degenerate Sink*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$



Behavior of Linear second Order Systems

Case 2: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{R}$, One can show that the state trajectories are given by

$$\begin{aligned} \dot{z}_1 &= z_{10}e^{\lambda t} + z_{20}te^{\lambda t} \\ \dot{z}_2 &= z_{20}e^{\lambda t} \end{aligned} \quad (3)$$

Assume $z_{20} \neq 0$. If $\lambda \neq 0$, from (3-b-) one gets

$$e^{\lambda t} = \frac{z_2(t)}{z_{20}} \Longrightarrow t = \frac{1}{\lambda} \ln\left(\frac{z_2(t)}{z_{20}}\right)$$

and using (3-a-) one obtains

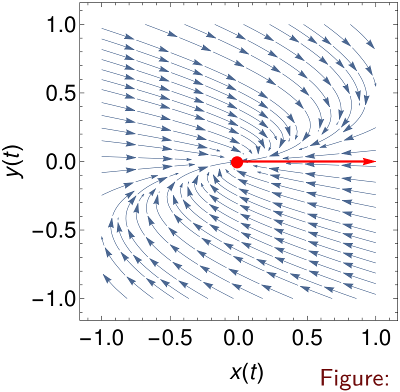
$$z_1(t) = z_{10}\frac{z_2(t)}{z_{20}} + \frac{1}{\lambda} \ln\left(\frac{z_2(t)}{z_{20}}\right) z_2(t)$$

Behavior of Linear second Order Systems

Case 2: $\lambda \neq 0$

The origin is called **stable/unstable improper node**

Only the z_1 axis is invariant.



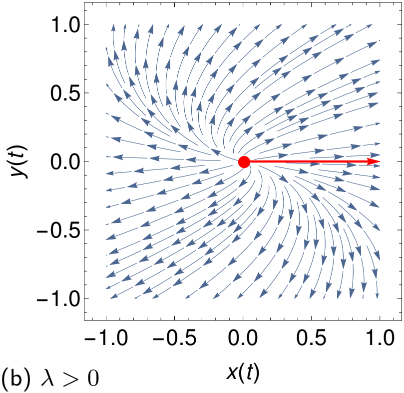


Figure: (a) $\lambda < 0$, (b) $\lambda > 0$

Behavior of Linear second Order Systems

Case 3: Complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm j\beta \in \mathbb{C}$
Let $\mathbf{v}_1 = u + jv$, $\mathbf{v}_2 = u - jv$ be the eigenvectors associated to the eigenvalues $\lambda_1 = \alpha + j\beta$, $\lambda_2 = \alpha - j\beta$. One has

$$A(u + jv) = (\alpha + j\beta)(u + jv) \qquad A(u - jv) = (\alpha - j\beta)(u - jv)$$

Summing and subtracting: $Au = \alpha u - \beta v \qquad Av = \beta u + \alpha v$

$$\Rightarrow T = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are the real eigenvectors of A associated with λ_1 and λ_2, respectively.
Defining the change of coordinates

$$r = \sqrt{z_1^2 + z_2^2} \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

we can write the dynamic equations in polar coordinates as

$$\dot{r} = \alpha r, \qquad \dot{\theta} = \beta$$

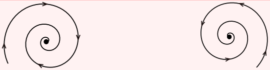
with solution

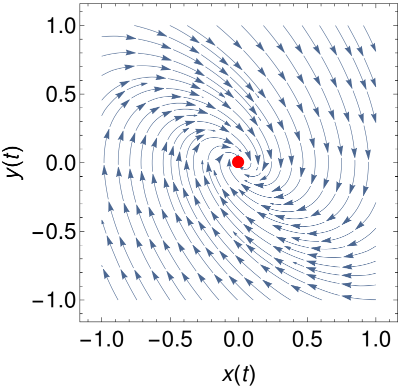
$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \theta_0 + \beta t$$

Behavior of Linear second Order Systems

Case 3a: $\alpha < 0$

The origin is called **Stable Focus**





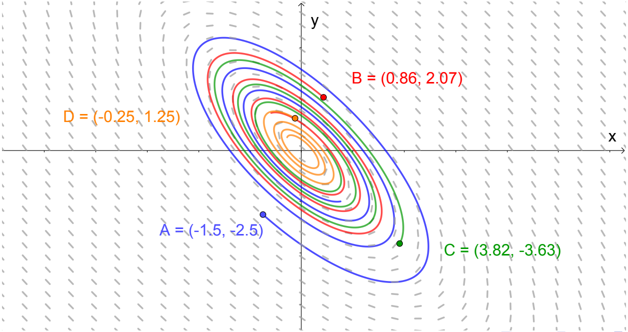
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= -2.2x - 2.9y \\ \dot{y} &= 2.9x + 2y \end{aligned}$$

eigenvalues : $\lambda_1 = -0.1 + 2j;\ \lambda_2 = -0.1 - 2j \implies$ *Stable Focus*

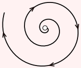
eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 2.9 \\ -2.1 - 2j \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 2.9 \\ -2.1 + 2j \end{pmatrix}$

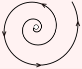


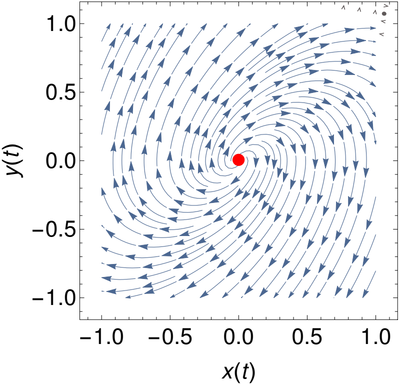
Behavior of Linear second Order Systems

Case 3a: $\alpha > 0$

The origin is called **Unstable Focus**



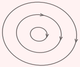


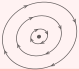


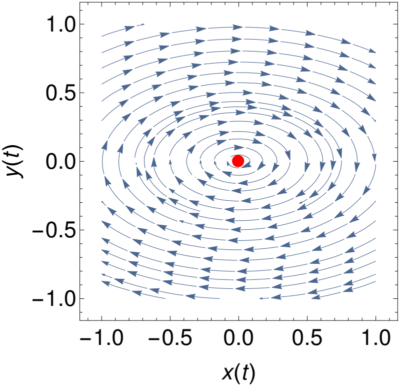
Behavior of Linear second Order Systems

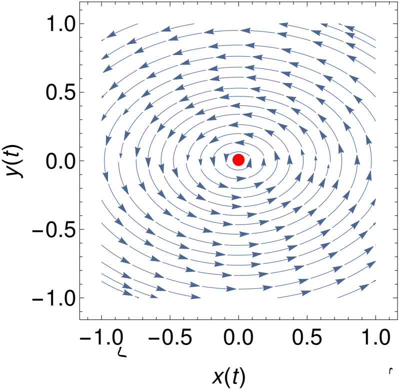
Case 3a: α

The origin is called **Center**









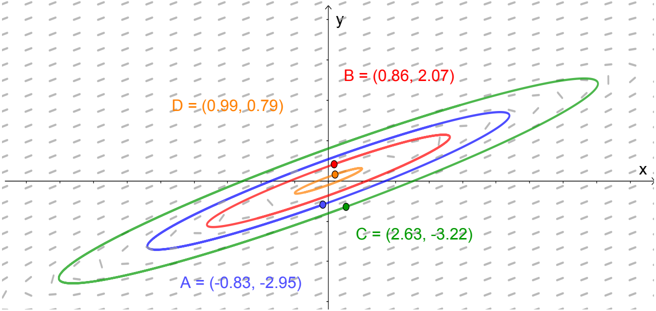
Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned}\dot{x} &= -3x + 10y \\ \dot{y} &= -x + 3y\end{aligned}$$

eigenvalues : $\lambda_1 = 0 + j;\ \lambda_2 = 0 - j \Longrightarrow$ *Center*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 10 \\ 3+j \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 10 \\ 3-j \end{pmatrix}$



Behavior of Linear second Order Systems

Phase Diagram All of these behaviors can be classified according to the trace T_r and the determinant Det of the matrix A. Recall that for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Find the eigenvalues of A :

$$det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - T_r\lambda + Det = 0$$

Thus the eigenvalues are

$$\begin{aligned} Tr(A) &\equiv a + d = \lambda_1 + \lambda_2 \Rightarrow \tfrac{1}{2}T_r(A) &= m \quad (\text{mean}) \\ Det(A) &\equiv ad - bc = \lambda_1\lambda_2 &= p \ (product) \\ \lambda_1, \lambda_2 &= m \pm \sqrt{m^2 - p} \end{aligned}$$

Behavior of Linear second Order Systems

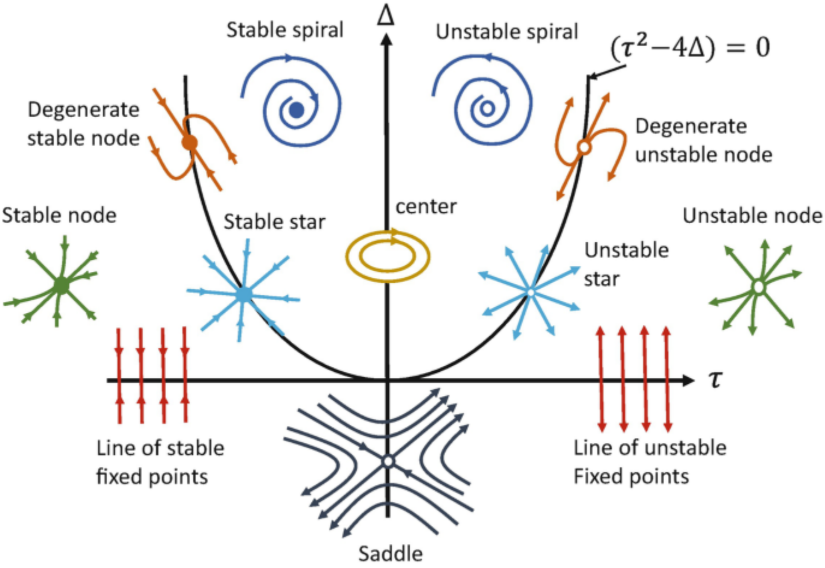
The values of (m, p) determine the equilibrium type.

- If $p < 0$, then the eigenvalues are real with opposite signs (**saddle node**).
- if $m^2 < p$, then the eigenvalues are complex with a real part (**spiral**: unstable if $m > 0$ and stable if $m < 0$).
- If $m = 0$ and $p > 0$, then the eigenvalues are purely imaginary (a **center**).
- $p > 0$ and $m^2 > p$ then the eigenvalues are real with the same sign (a **node**: stable if $m > 0$ and unstable if $m < 0$).

For linear system
The global qualitative behavior is determined by the type of equilibium point.

For nonlinear system Only local qualitative behavior in the vicinty of equilibrium point is determined by the type of equilibium point.

Behavior of Linear second Order Systems



Qualitative Behavior Near Equilibria

Given the nonlinear system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned} \qquad (4)$$

let us assume $x_e = (x_{1e}, x_{2e})$ is an equilibrium point of (4) i,e.,

$$f_1(x_{1e}, x_{2e}) = f_2(x_{1e}, x_{2e}) = 0$$

f_1, f_2 are continuously differentiable about (x_{1e}, x_{2e})
Since we are interested in trajectories near (x_{1e}, x_{2e}), define

$$x_1 = x_{1e} + \tilde{x}_1, \qquad x_2 = x_{2e} + \tilde{x}_2$$

$\tilde{x}_1$, $\tilde{x}_2$ are small perturbations form equilibrium point.
Expanding (4) into its Taylor series

$$\dot{x}_1 = \dot{x}_{1e} + \dot{\tilde{x}}_1 = \underbrace{f_1(x_{1e}, x_{2e})}_{0} + \left.\frac{\delta f_1(x)}{\delta x_1}\right|_{x_e} \tilde{x}_1 + \left.\frac{\delta f_1(x)}{\delta x_2}\right|_{x_e} \tilde{x}_2 + H.O.T$$

$$\dot{x}_1 = \dot{x}_{1e} + \dot{\tilde{x}}_1 = \underbrace{f_1(x_{1e}, x_{2e})}_{0} + \left.\frac{\delta f_1(x)}{\delta x_1}\right|_{x_e} \tilde{x}_1 + \left.\frac{\delta f_1(x)}{\delta x_2}\right|_{x_e} \tilde{x}_2 + H.O.T$$

For sufficiently small neighborhood of equilibrium points, H.O.T. are negligible

$$\begin{array}{rcl}\dot{\tilde{x}}_1 & = & a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 \\ \dot{\tilde{x}}_2 & = & a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2\end{array}, \quad a_{i,j} = \left.\frac{\delta f_i}{\delta x}\right|_{x_e}, \quad i = 1, 2.$$

The equilibrium point of the linear system is

$$\tilde{x} = A\tilde{x}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix} = \left.\frac{\delta f}{\delta x}\right|_{x_e}$$

Matrix $\frac{\delta f}{\delta x}$ is called Jacobian Matrix .

The trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearization about that point:

if the origin of the linearized state equation is a

- stable (unstable) node, or a stable (unstable) focus or a saddle point,

then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like a

- stable (unstable) node, or a stable (unstable) focus or a saddle point.

Qualitative Behavior Near Equilibria

Example

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^2 + 2\\ \dot{x}_2 &= 2(x_1^2 - x_2^2)\end{aligned}$$

Equilibrium points: $f(x_e) = 0$: $(-1,-1),\ (2,2),\ (1,-1),\ (-2,2)$

Linearization :

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -2x_1 & 1 \\ 4x_1 & -4x_2 \end{bmatrix}$$

Linearization around $(-1,-1)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(-1,-1)} = \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix}$$

$$\text{Eigenvalues} := \{3 \pm j\sqrt{3}\}$$

$\Rightarrow$ **Unstable focus** type of equilibrium.

Linearization around $\bar{x} = (2,2)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(2,2)} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\text{Eigenvalues} := \{-6 \pm 2\sqrt{3}\}$$

$\Rightarrow$ **stable node** type of equilibrium.

Qualitative Behavior Near Equilibria

Linearization around $\bar{x} = (-1, -1)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(-1,0)} = \begin{bmatrix} -2 & 1 \\ 4 & 4 \end{bmatrix}$$

; Eigenvalues : $= \lambda_1 = 1 + \sqrt{13} > 0$

$$\lambda_2 = 1 - \sqrt{13} < 0$$

$\Rightarrow$ **Saddle** type of equilibrium.

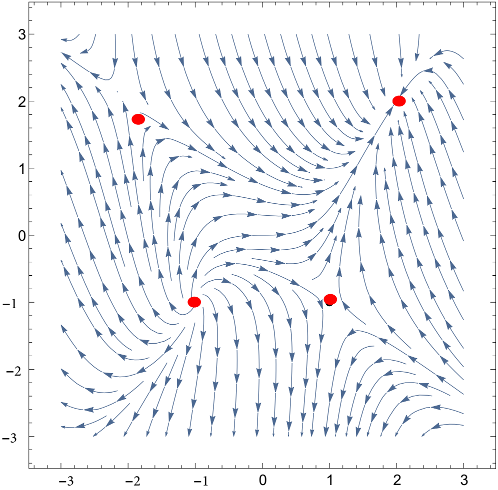
Linearization around $\bar{x} = (-2, 2)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(-2,2)} = \begin{bmatrix} 4 & 1 \\ -8 & -8 \end{bmatrix}$$

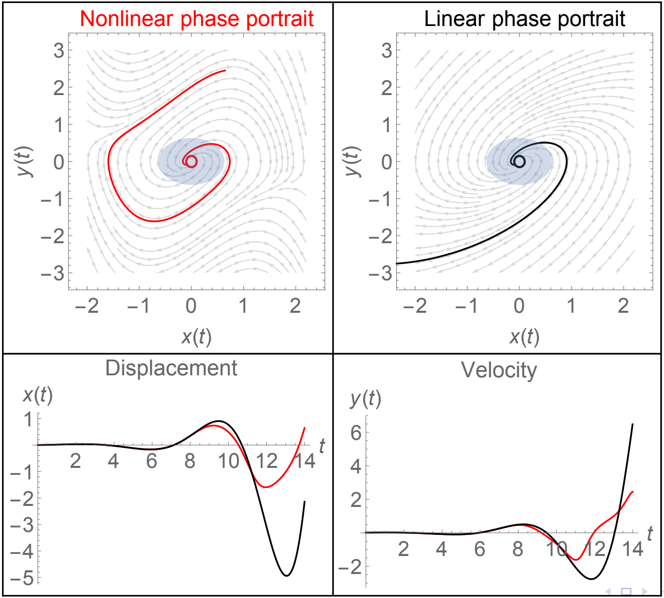
Eigenvalues : $= \lambda_1 = -2 + 2\sqrt{7} > 0$

$$\lambda_2 = -2 - 2\sqrt{7} < 0$$

$\Rightarrow$ **Saddle** type of equilibrium.

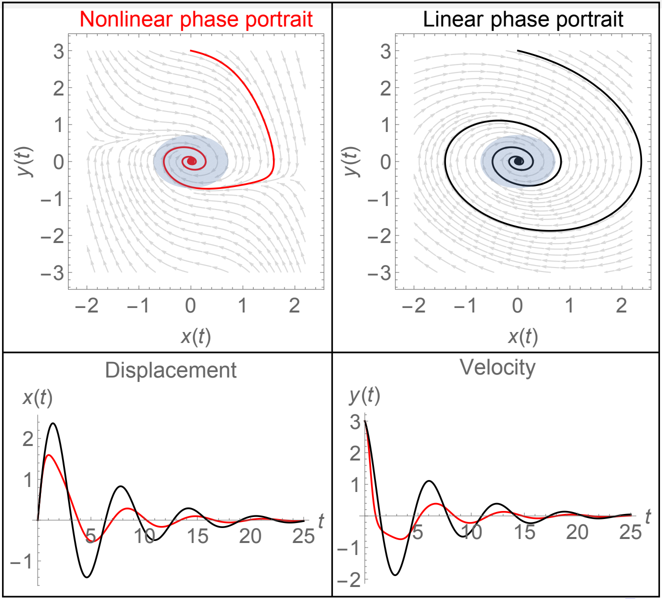


Qualitative Behavior Near Equilibria



Example: The Liénard equation (red) and its linearization (black). Parameter $\mu = 0.95$

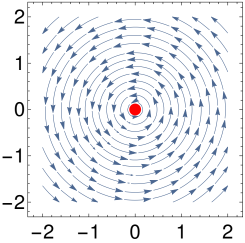
Qualitative Behavior Near Equilibria



Example: The Liénard equation (red) and its linearisation (black). Parameter $\mu = -0.35$

Example: ambiguous borderline case

$$\begin{cases} \dot{x}_1 = -x_2 + \underbrace{\mu x_1(x_1^2 + x_2^2)}_{\text{nonlinear terms}} \\ \dot{x}_2 = x_1 + \overbrace{\mu x_2(x_1^2 + x_2^2)} \end{cases} \qquad (5)$$



Fixed point : $(x_{1e}, x_{2e}) = (0,0)$.
Linearization :

$$J = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix}\Bigg|_{0,0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It is important to note that the linearized system does not depend on the control parameter μ.

Classification of the fixed point of the linearized system.

Trace of the system matrix is $T_r = 0$.

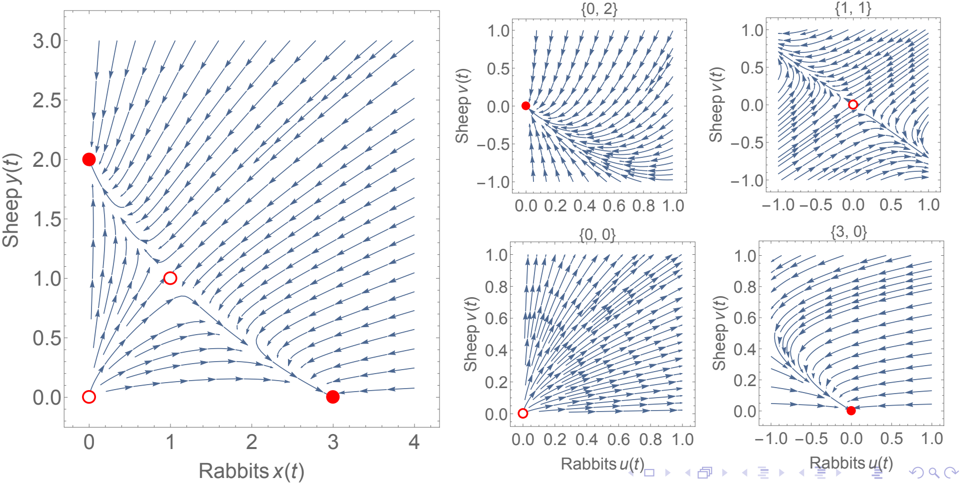
Determinant of the system matrix is $p = 1$.

The linear fixed point is a **centre**.

Example : The Lotka-Volterra competitive cohabitation model2 from ecology competitive cohabitation of rabbits and sheep. The model has the following form:

$$\begin{cases} \dot{x} = x(3-x) - 2xy \\ \dot{y} = y(2-y) - xy \end{cases} \tag{6}$$

where x and y are the sizes of rabbit and sheep populations, respectively.



Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Hyperbolic equilibrium point: linearization has no eigenvalues on the imaginary axis.

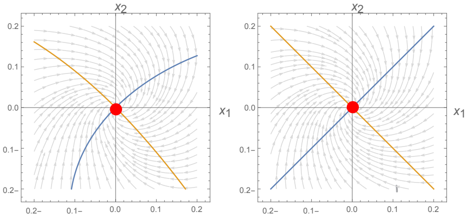
Hartman-Grobman Theorem:

If x_e is a hyperbolic equilibrium of a planar dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ then there is neighborhood U around x_e and a *homeomorphism*[1]

$$h : U \to \mathbb{R}^2$$

that maps the nonlinear trajectories in U to the linear trajectories in $\mathbb{R}^2$.

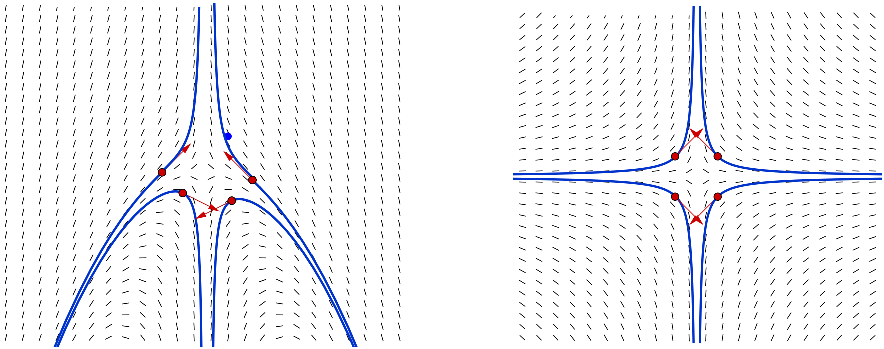
[1] homeomorphism: a continuous map with a continuous inverse (i.e. a change of coordinates)



Example: Consider the non-linear autonomous system

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + x_1^2 \end{aligned}$$

Equilibrium point : $\dot{x} = 0 \Rightarrow \bar{x} = (0,0)^T$.
Eigenvalues: $\lambda_1 = -1,\ \lambda_2 = 1 \Longrightarrow$ **saddle** type of equilibrium.



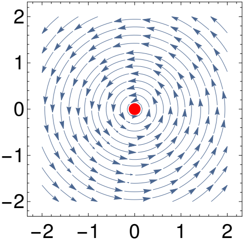
Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Example

$$\begin{aligned}\dot{x}_1 &= -x_2 + \mu x_1(x_1^2+x_2^2)\\ \dot{x}_2 &= x_1 + \mu x_2(x_1^2+x_2^2)\end{aligned} \tag{7}$$

There is only one equilibium point at (0,0), and the linearized system at this point is

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \lambda_{1,2} = \{\pm J\}$$



$\Longrightarrow$ the equilibrium point is **center**. Since this equilibrium point is *non-hyperbolic* $\Longrightarrow$ No conclusion about the behavior of the nonlinear system near (0,0)

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

System (7) is analysed in polar coordinates. Usually a direct coordinate transform

$$\left\{\begin{array}{lcl} x_1 & = & r\cos\theta \\ x_2 & = & r\sin\theta \end{array}\right. ,\left\{\begin{array}{lcl} r & = & \sqrt{x_1^2+x_2^2} \\ \theta & = & \tan\left(x_2/x_1\right) \end{array}\right. \tag{9}$$

where $r = r(t)$ and $\theta = \theta(t)$, is used. We are searching a system in the form:

$$\left\{\begin{array}{lcl} \dot{r} & = & f_1(r,\theta) \\ \dot{\theta} & = & f_2(r,\theta) \end{array}\right. \tag{10}$$

where functions $f_1(r,\theta)$ and $f_2(r,\theta)$ are to be determined. We are interested in temporal dynamics of (9)

$$\left\{\begin{array}{lll} 2r\dot{r} & = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 & \Rightarrow \dot{r} = \dfrac{x_1 f_1(x_1,x_2) + x_2 f_2(x_1,x_2)}{r} \\ \sec^2\theta\dot{\theta} & = (1+\tan^2\theta)\dot{\theta} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} & \Rightarrow \dot{\theta} = \dfrac{x_1 f_2(x_1,x_2) - x_2 f_1(x_1,x_2)}{x_1^2+x_2^2} \end{array}\right.$$

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

After developing (11). The system (7) has been represented in polar coordinates. Resulting decoupled equations

$$\begin{aligned} \dot{x}_1 &= -x_2 + \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + \mu x_2(x_1^2 + x_2^2) \end{aligned} \Longrightarrow \begin{cases} \dot{r} = \mu r^3 \\ \dot{\theta} = 1 \end{cases}$$

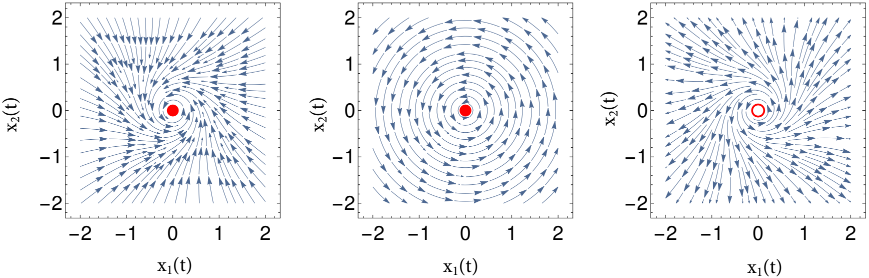


Figure: $\mu < 0$, $\mu = 0$, $\mu > 0$

Non-existence of Periodic Orbits

Bendixson criterion gives a sufficient condition for detecting the absence of periodic orbits for second-order systems (Limit cycles or neutrally stable cycles).

Bendixson criterion:

For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$$

If $div(f) = \nabla . f(x) = [\partial/\partial x_1 \quad \partial/\partial x_2] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

Example 1: $\dot{x} = Ax, x \in \mathbb{R}^2$ can have periodic orbits only if div f= trace(A)=0.,

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$$

Unless $trace(A) = 0 \Longrightarrow$ non periodic orbits.

Example 2:

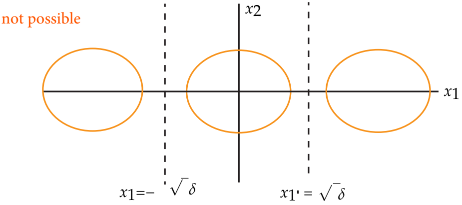
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2, \qquad \delta > 0 \end{aligned}$$

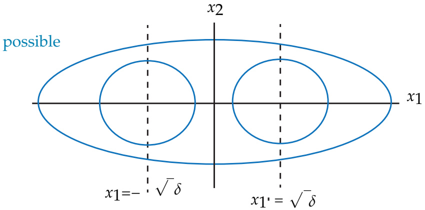
$$\nabla . f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

$$\nabla . f(x) = 0, \quad \text{then} \quad x_1 = \pm\sqrt{\delta}$$

Therefore, no periodic orbit can lie entirely in the region

$$x_1 \in]-\infty, -\sqrt{\delta}[, \quad]-\sqrt{\delta}, \sqrt{\delta}[, \quad]\sqrt{\delta}, +\infty[$$





Periodic Orbits in the Plane

Let $\phi(t, x_0)$ denotes the solution of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Definition: Invariant sets

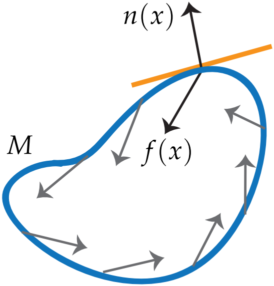
A set $M \subseteq \mathbb{R}^n$ is **positively invariant** if, for each $x_0 \in M$, $\phi(t, x_0) \in M$ for all $t \geq 0$.

Theorem

If $V : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^1$ and $M = \{x : V(x) \leq c\}$, then M is invariant if

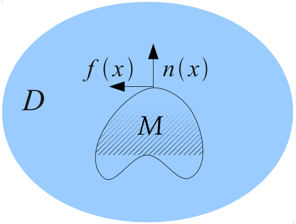
$$f(x).\nabla V(x) \leq 0 \quad \forall x : V(x) = c$$

i.e. if x is on the boundary of M, then the vector $f(x)$ points into M.

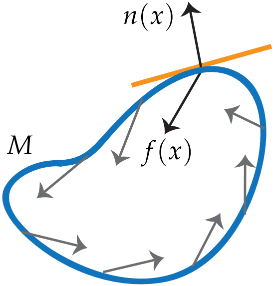


Non-existence of Periodic Orbits

Example: Consider a closed orbit $\dot{x} = f(x)$.
$f(x)$ is tangential to the trajectory x. Along this closed trajectory : $f^T(x).\vec{n} = 0$.
The interior of any closed trajectory is a positively invariant set.



Example:
If $f(x)^T.\vec{n} \leq 0$ then M is positively invariant.
$\vec{n}$: outward normal on a boundary of M.
Along boundary of M:for all $x \in \partial M \Rightarrow [f(x)]^T.\vec{n} \leq 0$.



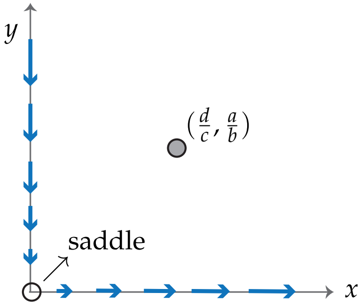
Periodic Orbits in the Plane

Example: Predator-prey model

$$\begin{aligned} \text{prey}: \quad \dot{x}_1 &= (a - bx_2)x_1 \\ \text{predator}: \quad \dot{x}_1 &= (cx_1 - d)x_2 \end{aligned}$$

a,b,c,d positive parameters.

Equilibrium points : $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} d/c \\ a/b \end{bmatrix}.$



Clearly $[f(x)]^T.\vec{n} = 0$ along the boundary of $M = \{x_1 \geq 0, x_2 \geq 0\}$, which means the first quadrant M is positively invariant.

Linearization around $\bar{x} = (0,0)$: $A = \frac{\partial f}{\partial x}\Big|_{\bar{x}} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}.$

$\lambda_1 = a > 0$ and $\lambda_2 = -d < 0 \Longrightarrow$ **saddle** type of equilibrium.

Periodic Orbits in the Plane

Example 2:

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

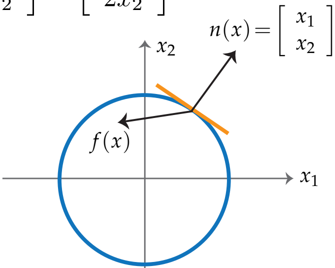
Show that $B_r := \{x \in \mathbb{R}^2 / x_1^2 + x_2 \leq r^2\}$ is positively invariant for sufficiently large r (to be determined). We want to calculate $[f(x)]^T n(x)$

$$V(x) = x_1^2 + x_2^2 = r^2 \Rightarrow n(x) = \nabla V(x) = \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\begin{aligned}[f(x)]^T.n(x) &= = f_1 \tfrac{\partial V}{\partial x_1} + f_2 \tfrac{\partial V}{\partial x_2} \\ &= -2(x_1^2 + x_2^2)^2 + 2x_1^2 + 2x_2^2 - 2x_1 x_2\end{aligned}$$

$$-2x_2 x_2 \leq x_1^2 + x_2^2 \qquad \text{(completion of squares).}$$

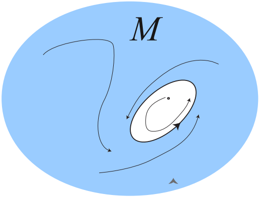
Therefore $[f(x)]^T.n(x) \leq -2r^2(r^2 - 3/2)) \leq 0$ if $r^2 \geq 3/2$.

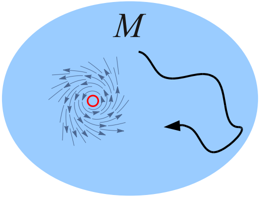


Existence Theorem of Limit Cycle

Poincaré-Bendixson Theorem:

Let M be a compact (closed and bounded) set in $\mathbb{R}^2$, which is positively invariant for $\dot{x} = f(x)$, $x \in \mathbb{R}^2$. If M does not contain an equilibrium point, then it contains a periodic orbit.

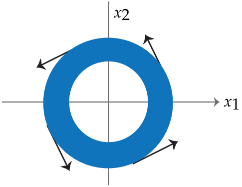




The "no equilibrium condition" in PB Theorem can be relaxed as: " M can have one equilibrium point which is either an unstable focus or an unstable node, then there is a periodic orbit.

Example: harmonic oscillator

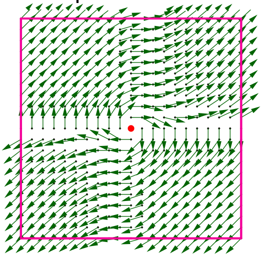
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



For any $R > r > 0$, the ring $\{x : r^2 \leq x_2 + x_2 \leq R^2\}$ is compact, invariant and contain no equilibria.

$[f(x)]^T.n(x) = 0$ everywhere and PB Theorem state there exists a periodic orbit (or more) in M.

Example:2



$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_2^2 \\ \dot{x}_1 &= -x_1 + x_1^2 x_2 \end{aligned}$$

Linearization around the equilibrium at $\bar{x} = (0\ 0)^T$ yields

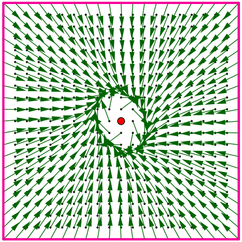
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which exhibits a continuum of periodic solutions. However, for this nonlinear system, we have

$$\nabla.f(x) = x_1^2 + x_2^2 > 0, \quad \forall x \neq 0$$

Hence, Bendixson theorem leads to the conclusion that this dynamical system has no nontrivial periodic solutions.

Existence Theorem of Limit Cycle

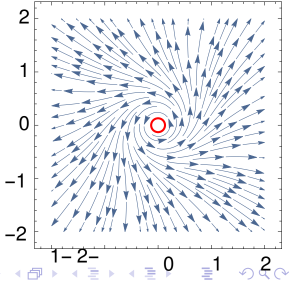


Example:3

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

$[f(x)]^T.n(x) \leq 0$ iff $r^2 = x_1^2 + x_2^2 > 3/2$. i.e. $B_r = x \in \mathbb{R}^2 / x_1^2 + x_2^2 \leq r^2$ is positively invariant $r \geq \sqrt{3/2}$ but contains the equilibrium $x_e = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{x_e=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix},\ \lambda_{1,2} = 1 \pm j\sqrt{2},\ \text{unstable focus}$$



Therefore, B_r must contain a periodic orbit.

Limit Cycle: Stable Limit Cycle

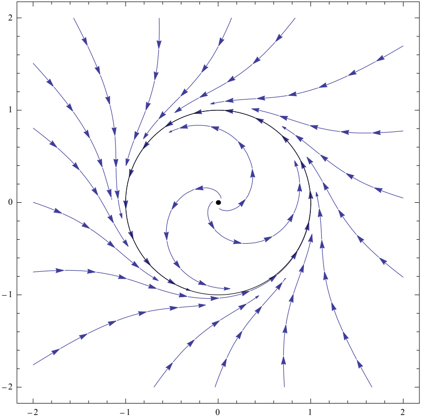
All trajectories in the vicinity of the limit cycle converges to it as $t \to \infty$

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_1 &= -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

$$\Longrightarrow \begin{cases} \dot{r} = -r(r^2 - 1) \\ \dot{\theta} = -1 \end{cases} \quad (12)$$

if $r > 1 \to \dot{r} > 0$	converging
if $r > 1 \to r < 0$	converging
if $r = 1 \to \dot{r} = 0$	remaining



Limit Cycle: Unstable Limit Cycle

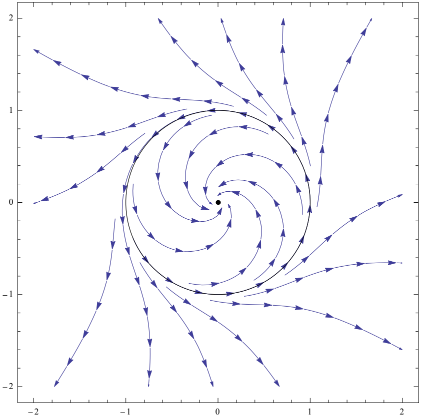
All trajectories in the vicinity of the limit cycle diverges from it as $t \to \infty$

Example

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_1 &= -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

$$\Longrightarrow \begin{cases} \dot{r} = r(r^2 - 1) \\ \dot{\theta} = -1 \end{cases}$$

if $r < 1 \to \dot{r} < 0$	diverging
if $r > 1 \to r > 0$	diverging
if $r = 1 \to \dot{r} = 0$	remaining



Limit Cycle: half-stable Limit Cycle

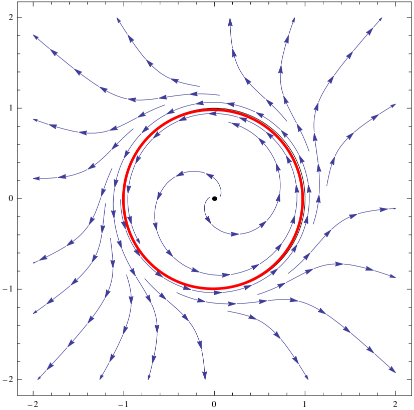
Some of the trajectories in the vicinity of the limit cycle converges to it, while others diverge from it as $t \to \infty$

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_1 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{aligned}$$

$$\Longrightarrow \begin{cases} \dot{r} &= -r(r^2 - 1)^2 \\ \dot{\theta} &= -1 \end{cases}$$

if $r < 1 \to \dot{r} < 0$	diverging
if $r > 1 \to r < 0$	converging
if $r = 1 \to \dot{r} = 0$	remaining



Bifurcation: Holf Bifurcation

Example: Supercritical Hopf bifurcation

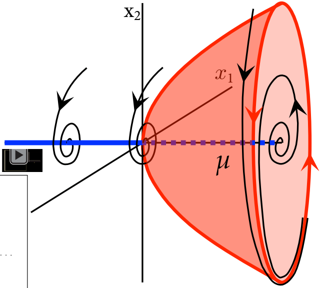
$$\begin{cases} \dot{x}_1 &= -x_2 + x_1(\mu - x_1^2 - x_2^2) \\ \dot{x}_2 &= +x_1 - x_2(\mu - x_1^2 - x_2^2) \end{cases} \Longrightarrow \begin{cases} \dot{r} &= r(\mu - r^2) \\ \dot{\theta} &= 1 \end{cases} \quad (15)$$

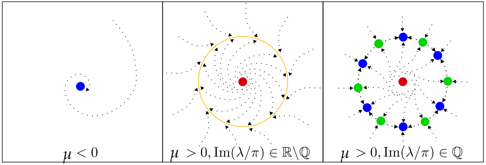
Equilibrium points : $r(\mu - \bar{r}^2) = 0.$
Note that a positive equilibrium for the r subsystem means a limit cycle in the (x_1, x_2) plane.

$\mu < 0$: stable equilibrium at $r = 0$.

$\mu > 0$ unstable equilibrium point ar $r = 0$ and stable limit cycle at $r = \sqrt{\mu}$.

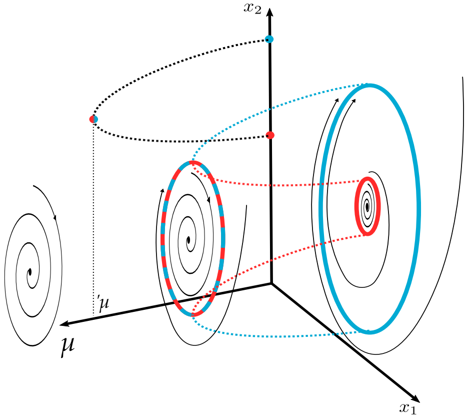
The origin loses stability at $\mu = 0$ and a stable limit cycle emerges.





Bifurcation: Holf Bifurcation

In Supercritical Hopf bifurcation by increase of μ near zero, the stable equilibrium point becomes unstable but a stable limit cycle appears. Hence, this is a Safe bifurcation.



Chapter 3

Existence and Uniqueness of Solutions

To be a useful mathematical model of a physical system, the state equation

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

must have the following properties:

- Existence of solution.
- Uniqueness of solution.
- Continuous dependence of solution on initial conditions.
- Continuous dependence of solution on parameters.

The first question ask is the **Cauchy problem:**

Definition (Cauchy Problem)

The Cauchy problem is to find a unique, continuous $x : [0, t_f] \to \mathbb{R}^n$ for some t_f such that $\dot{x} = f(t, x)$ for all $t \in [0, t_f]$.

Existence of Solutions

There exist many system for which no solutions exists or for which a solution only exists over a finite time interval.

Example Consider

$$\dot{x} = x^2, \quad x(0) = x_0$$

We have

$$\frac{dx}{x^2} = dt, \; -\frac{1}{x} = t + C \Rightarrow C = -\frac{1}{x_0}$$

The solution

$$x(t) = \frac{x_0}{1 - x_0 t}$$

If $x_0 > 0$, the maximal solution is defined on $(-\infty, 1/x_0[$.

The escape time $t_e = \frac{1}{x_0}$

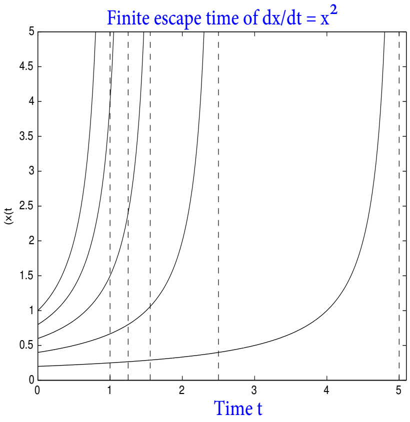


Figure : Simulation of $x^{\cdot} = x^2$ for several

Non-Uniqueness

Example: A classical example of a system without a **unique** solution is

$$\dot{x} = x^{1/3}, \qquad x(0) = 0$$

For the given initial condition, it is easy to verify that

$$x(0) = 0 \qquad \text{and} \qquad x(t) = \left(\frac{2t}{3}\right)^{3/2}$$

both satisfy the differential equation.

Existence and Uniqueness of Solutions

Theorem : Local Existence and Uniqueness

Let $f(t,x)$ be a piece- wise continuous function in t and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \qquad (1)$$

$\forall x, y \in \mathcal{B}(x_0, r) = \{x \in \mathbb{R}^n | \|x - x_0\| \leq r\}$; $\forall t \in [t_0, t_1]$. Then there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

- If $f(x)$ is continuous (C^0) then a solution exists, but C^0 is not sufficient for uniqueness.
- Sufficient condition for uniqueness: "Lipschitz continuity" (more restrictive than C^0)$\Longrightarrow$ Lipschitz is stronger than continuity.

Existence and Uniqueness of Solutions

Example 1: $\dot{x} = x^{1/3}, x(0) = 0.$
For $x_0 = 0$, we have two solutions:

$$x(t) \equiv 0, \quad \text{and} \quad x(t) = \left(\frac{2}{3}t\right)^{\frac{2}{3}}.$$

This function is not Lipshitz. The gradient becomes infinite at $x = 0$.
Example 2: $f(x) = x^2 \rightarrow x(t) = \frac{x_0}{1-x_0 t}$

$$|f(y) - f(x)| = |y^2 - x^2| = |(x+y)(x-y)| \leq \underbrace{|(x+y)|}_{L(x,y)}.|x-y|$$

Example 3: $f(x) = x^3 \rightarrow x(t) = \frac{x0}{\sqrt{1-2x_0^2 t}}$. Interval of existence: $[1, 1/2x_0^2]$.

$$|f(y) - f(x)| = |y^3 - x^3| = |(y^2 + xy + x^2)(x-y)| \leq \underbrace{|(y^2 + xy + x^2)|}_{L(x,y)}.|x-y|$$

$f(x) = x^2$ and $f(x) = x^3$ are boyth locally Lipshitz but not globally Lipshitz.

Existence and Uniqueness of Solutions

If $f(.)$ is differentiable continuously differentiable (C^1) then it is locally Lipschitz.

Example 1:

$$f(x) = x^2 \Rightarrow \partial f/\partial x = 2x.$$
$$f(x) = x^3 \Rightarrow \partial f/\partial x = 3x^2.$$

Example 2:

$$f(x) = x^{1/3} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{3}x^{-2/3} \Rightarrow \text{Not continuous at } 0.$$

$$f(x) = 2\sqrt{x} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{\sqrt{x}} \Rightarrow \text{Not continuous at } 0.$$

f(x)=sat(x)
Not differentiable at $x \pm 1$ but locally Lipschitz:
|sat(x)-sat(y)|≤ |x-y|,   L=1.

Global Existence and Uniqueness

Let $f(t,x)$ be piece-wise continuous in t over the interval $[t_0, t_1]$ and globally Lipschitz in x. Then

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0$$

has a unique solution over $[t_0, t_1]$.

Example : sat(.) is globally Lipschitz. x^3 is not globally Lipschitz.

1. $f(.)$ is C^0 $\Rightarrow$ existence of solution $x(t)$ on finite interval $[0, t_f)$.

2. $f(.)$ locally Lipschitz $\Rightarrow$ existence and uniqueness on $[0, t_f)$.

3. $f(.)$ globally Lipschitz $\Rightarrow$ existence and uniqueness on $[0, \infty)$.

Continuous Dependence on Initial Conditions and Parameters

Theorem:(Continuous dependence on initial conditions)

Let $x(t), y(t)$ be two solutions of $\dot{x} = f(t, x)$ starting from x_0 and y_0 and remaining in a set with Lipschitz constant L on $[0, T]$. Then, for any $\epsilon > 0$, there exists $\delta(\epsilon, T) > 0$ such that :

$$\|x_0 - y_0\| \leq \delta \Longrightarrow \|x(t) - y(t)\| \leq \epsilon, \forall t \in [0, T]$$

Example: 2nd order LTI system with **saddle** type of equilibrium point. This example demonstrate that it is impossible to expect a small difference in trajectory for a small difference in initial conditions.

Lipshitz continuity of $f \Longrightarrow$ continuity of solutions with respect to initial conditions.

Continuous Dependence on Initial Conditions and Parameters

Continuous dependence on parameters

The previous Theorem also shows continuous dependence on parameter μ in f(t, x, μ) if we rewrite the system equations as:

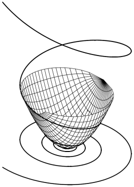
$$\begin{aligned}\dot{x} &= f(t, x, \mu) \\ \dot{\mu} &= 0\end{aligned} \qquad X = \begin{bmatrix} x \\ \mu \end{bmatrix} \qquad \dot{X} = F(t, X) = \begin{bmatrix} f(t, x, \mu) \\ 0 \end{bmatrix}$$

Chapter 4: Lyapunov Stability Theory

Stability

Stability theory is divided into three parts:

- Stability of equilibrium points.
- Stability of periodic orbits
- Input/output stability



Alexander Mikhailovich Lyapunov (1957-1918)

- Russian mathematician and physicist.
- Known for his development of the stability theory of dynamical systems.

> If the total energy is dissipated, then the system must be stable.

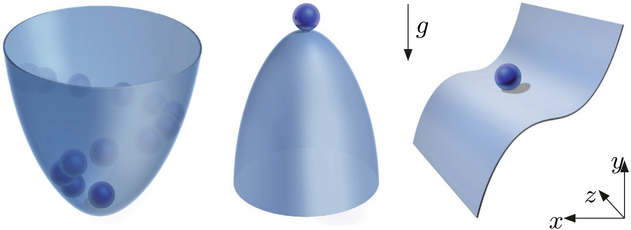
Stability of Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \qquad (1)$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a *locally Lipschitz* map from a domain $\mathcal{D} \subset$ into $\mathbb{R}^n$. Suppose that the system (1) has an equilibrium point $\bar{x} \in \mathcal{D}$, i.e., $f(\bar{x}) = 0$.

Without loss of generality and the simplicity of notation, we assume the equilibrium is located at the origin.



Autonomous Systems : stability of an equilibrium state

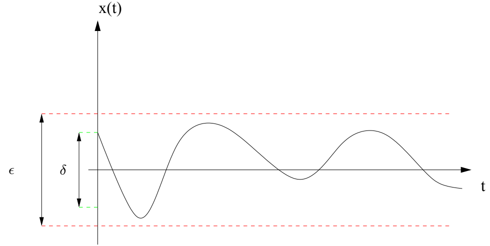
Stability in the sense of Lyapunov

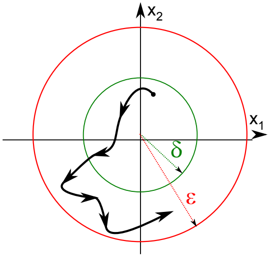
The equilibrium point $\bar{x} = 0$ of $\dot{x} = f(x)$ is

stable, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon, \forall t \geq 0.$$

unstable, if it is not stable.





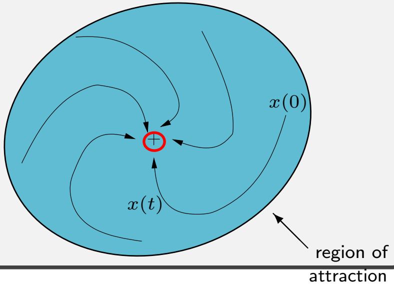
Autonomous Systems : stability of an equilibrium state

"Stability is a property of the equilibrium, not of the system"

Stability of the equilibrium is equivalent to stability of the system only when there exists only one equilibrium (e.g., linear systems). In this case **stability** $\equiv$ **global stability.**

The region of **attraction** of the equilibrium point $\bar{x} = 0$ of (1) is is the set of all initial conditions $x(0)$ for which

$$x(t) \to 0 \quad \text{as} \quad t \to \infty.$$



Autonomous Systems : stability of an equilibrium state

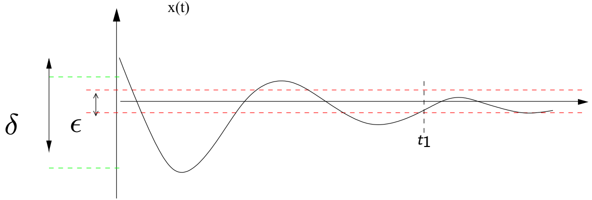
Stability in the sense of Lyapunov

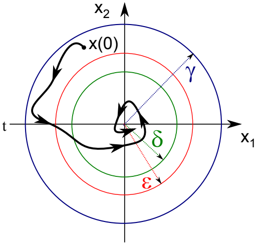
The equilibrium point $\bar{x} = 0$ of (1) is

attractive, if there exist δ such that :

$$\|x(0)\| < \delta \Longrightarrow \lim_{t \to \infty} x(t) = 0.$$

local asymptotically stable (a.s) if it is **stable** and **attractive**





Autonomous Systems : stability of an equilibrium state

The equilibrium point $\bar{x} = 0$ of (1) is

- **exponentially stable** (e.s), if there exists $\alpha >, \beta > 0$ and $\delta > 0$ such that

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| \leq \beta \|x(0)\| e^{-\alpha t}, \forall t \geq 0$$

- **globally asymptotically stable** (g.a.s) if it is stable and globally attractive, i.e $\lim_{t\to\infty} x(t) = 0$. for all $x(0) \in \mathbb{R}^n$.
- **globally exponentially stable**(g.e.s), for all $x(0) \in \mathbb{R}^n$, there exists $\beta > 0$ and $\alpha > 0$ such that

$$\|x(t)\| \leq \beta \|x(0)\| e^{-\alpha t}, \forall t \geq 0$$

exponential stability is a special case of asymptotic stability.

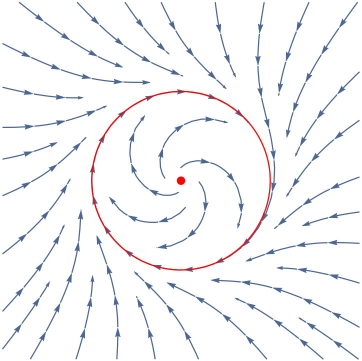
Stability, AS, ES: local concepts (for x(0) sufficiently close to ")

Remarks on stability

Attractively does not imply asymptotic stability

Example: Consider the second-order system with state variables x_1 and x_2 whose dynamics are most easily described in polar coordinates via the equations

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= \sin^2(\theta/2) \end{aligned} \quad (2)$$



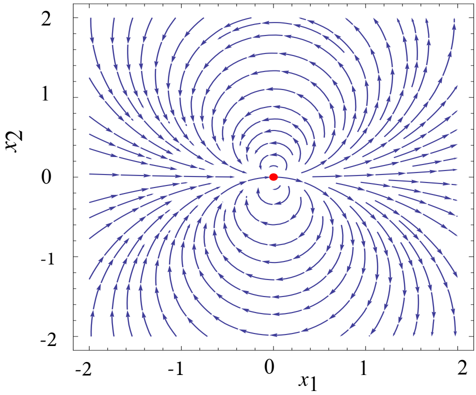
where $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$, $\theta \in [0, 2\pi[$.
This system has two equilibrium points : (r^*, θ^*)=(0,0) and (r^*, θ^*)=(1,0).
The fixed point at zero is clearly unstable.
The fixed point with $r^* = 1$ attracts all other trajectories, but it is not stable by any of our definitions.

Remarks on stability : Chaotic attractor

Example

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{aligned}$$

All trajectories converge to $x_e = 0$ but x_e is not stable.



Convergence by itself does not imply Stability

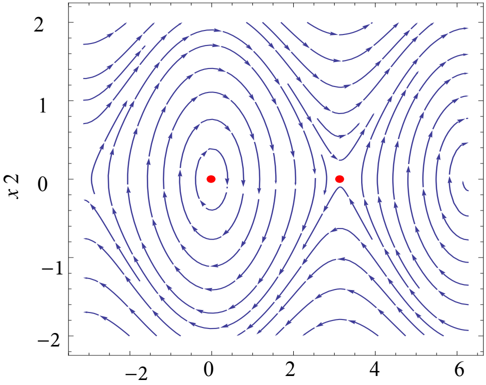
Remarks on stability

There exist Lyapunov-stable sets that are not attractors.

Example: stable system but not attractor

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1)\end{aligned}$$

Show that an equilibrium point $x^- = [2k\pi, 0]^T$ is stable but not .attractor



Remarks on stability

Example:

$$\dot{x} = -x^2 \Longrightarrow x(t) = \frac{x(0)}{1 + tx(0)}$$

$\bar{x} = 0$ is GAS (but not GES).

Problems

- How to check the stability properties of $\bar{x} = 0$ WITHOUT computing state trajectories ?
- if $\bar{x} = 0$ is A.S how to compute a region of attraction ?
- The analysis of the linearized system around $\bar{x} = 0$ MIGHT allow one to check local stability. How to proceeed when no conclusion can be drawn using the linearized system ?

Need of a more complete approach: **Lyapunov direct method**

Remarks on stability

Example

$$\dot{x} = ax - x^5$$

$\bar{x} = 0$ is an equilibrium state.
Linearized system: $\dot{x} = ax$

$a < 0 \Longrightarrow \bar{x} = 0$ is A.S

$a > 0 \Longrightarrow \bar{x} = 0$ is unstable

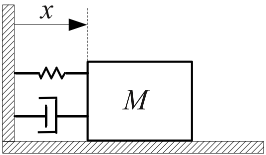
a=0 ?

For $a = 0$ one has $\dot{x} = -x^5$ and with the Lyapunov direct method one can show that $\bar{x} = 0$ is AS.

One way to check **stability** is to **plot** trajectories and see what is going on. This approach has limited utility because it requires solution to differential equation (which may difficult to solve!)

Lyapunov direct method

Example



Model

$$M\ddot{x} = \underbrace{-b\dot{x}|\dot{x}|}_{\text{NL damping}} - \underbrace{(k_0 x + k_1 x^3)}_{\text{NL elasticforce}}$$

Defining $x_1 = x$ and $x_2 = \dot{x}_1$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{M}x_2|x_2| - \frac{k_0}{M}x_1 - \frac{k_1}{M}x_1^3 \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{is stable/AS/ES}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} - 3\frac{k_1}{M}x_1^2 & -\frac{2b}{M}x_2 \text{sgn}(x_2) \end{bmatrix} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} & 0 \end{bmatrix}$$

Eigenvalues: $\lambda_{1,2} = \pm j\sqrt{k_0/M}$.

No conclusion on $\bar{x} = 0$ using the linearized system.

Lyapunov direct method

Consider the total energy of the system:

$$V(x) = \underbrace{\frac{1}{2}Mx_2^2}_{kinetic} + \underbrace{\frac{1}{2}k_0x^2 + \frac{1}{4}k_1x_1^4}_{potential}$$

Remark: zero energy$\Leftrightarrow$x$_1 = x_2 = 0$ (equilibrium state)

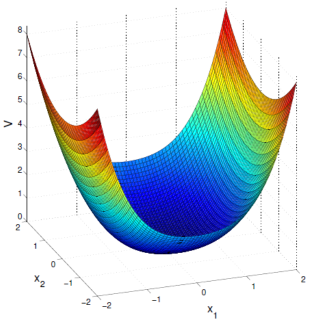
Instantaneous energy change:

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}.\frac{dx}{dt} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \left(k_0x_1 + k_1x_1^3\right)\dot{x}_1 + Mx_2\dot{x}_2 \\ &= (k_0x_1 + k_1x_1^3)x_2 + Mx_2(-\frac{b}{M}x_2|x_2| - \frac{k_0}{M}x_1 - \frac{k_1}{M}x_1^3) = -bx_2^2|x_2| \end{aligned}$$

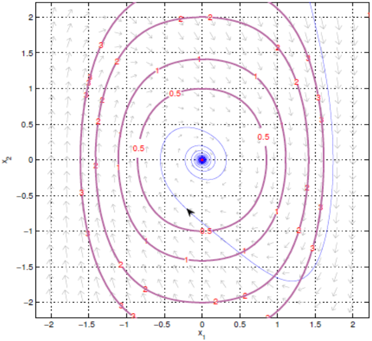
$-bx_2^2|x_2| \leq 0$ independently of $x(0)$ $\Rightarrow$ the energy can only decrease with time independently of $x(0)$

Lyapunov direct method: a first example

Energy V(x)



Phase plane



Energy is a "measure" of the distance of x from the origin

- if it can only decrease, then $\bar{x} = 0$ should be stable.

Lyapunov direct method is based on energy-like functions $V(x)$ and the analysis of the function

Derivative along the trajectory

Definition: Lyapunov Function

Let $V : \mathcal{D} \to \mathbb{R}$ be a *continuously differentiable* function defined in a domain $D \in \mathbb{R}^n$ that contains the origin. The derivative of V along the trajectory (solution) of $\dot{x} = f(x)$ denoted by $\dot{V}(x)$ is given by

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \nabla V.f(x)$$

$$= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

Lyapunov stability theory

Definition: Positive Definite Functions

A function $V : \mathcal{D} \to \mathbb{R}$ is *positive semi definite* in $\mathcal{D}$ if

(i). $V(x) = 0$ if and only if $x = 0$.

(ii). $V(x) \geq 0$, $\forall x$in $\mathcal{D} - \{0\}$.

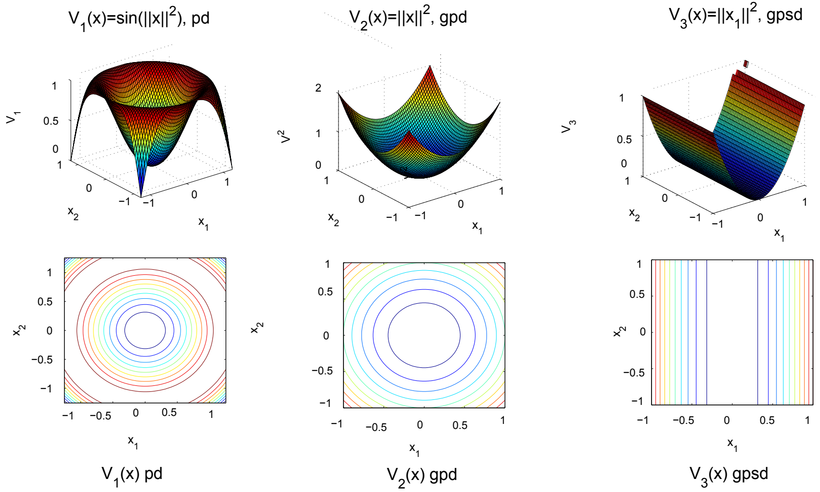
A function $V : \mathcal{D} \to \mathbb{R}$ is *positive definite* in $\mathcal{D}$ if

(ii'). $V(x) > 0$, $\forall x$in $\mathcal{D} - \{0\}$.

A function $V : \mathcal{D} \to \mathbb{R}$ is *negative definite* in $\mathcal{D}$ if $-V$ is positive definite (semi definite).

Lyapunov stability theory

Examples of positive definite functions



Lyapunov Stability Theorem

Theorem

1. Let $\bar{x} = 0$ be an equilibrium for $\dot{x} = f(x)$ and $\mathcal{D} \in \mathbb{R}^n$ be a domain containing $\bar{x} = 0$. If there exists a *continuously differentiable* function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \in \mathcal{D} - \{0\} \quad \text{(positive definite)}$$

and

$$\dot{V}(x) := \nabla V^T(x) f(x) \leq 0 \quad \forall x \in \text{in } \mathcal{D} \quad \text{(negative semidefinite)}$$

then, $\bar{x} = 0$ is **stable**.

2. If $\dot{V}(x) < 0$, $\forall x \in \mathcal{D} - \{0\}$ (negative definite)

then, $\bar{x} = 0$ is **asymptotically stable**.

3. If, in addition, $\mathcal{D} = \mathbb{R}^n$, and

$$\|x\| \to \infty \Longrightarrow V(x) \to \infty. \quad \text{(radially unbounded)}$$

then $\bar{x} = 0$ is **globally asymptotically stable.**

Lyapunov Stability Theorem

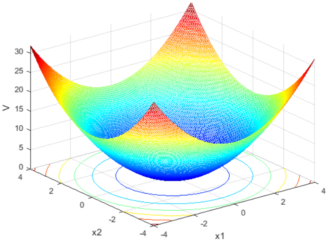
Example

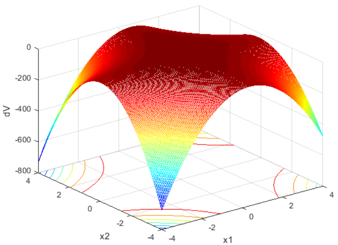
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= x_1 - x_2 - x_1^2 x_2\end{aligned}$$

Study the stability of the equilibrium state $\bar{x} = 0$.
Take the Lyapunov function $V(x) = x_1^2 + x_2^2$ (positive definite in $\mathbb{R}^2$)

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = 2x_1(-x_1 + x_2^2) + 2x_2(x_1 - x_2 - x_1^2 x_2) \\ &= -(x_1 - x_2)^2 - x_2^2(1 - x_1)^2 - x_1^2(1 + x_2^2)\end{aligned}$$

$\dot{V}(x)$ is negative definite $\Longrightarrow$ the system is stable





Lyapunov Stability Theorem

Q.why is radially unboundedness required for G.A.S ?

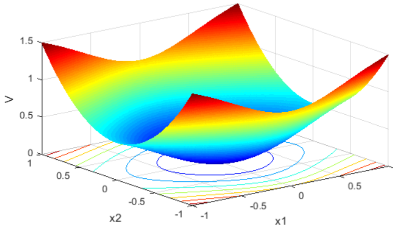
Example: Consider the following *positive definite function*

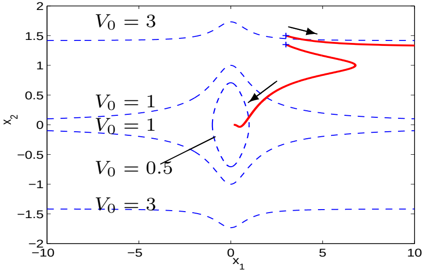
$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

if $x = (x_1, 0)$ then $\|x\| \to \infty \quad as \quad \mathrm{x}_1 \to \infty$.

But $V(x)$ will approach 26 (not radially unbounded)

$$\lim_{x_1 \to \infty} V(x) = \lim_{x_1 \to \infty} \frac{1 + x_1^2}{x_1^2} + (5)^2 = 26 \neq +\infty$$

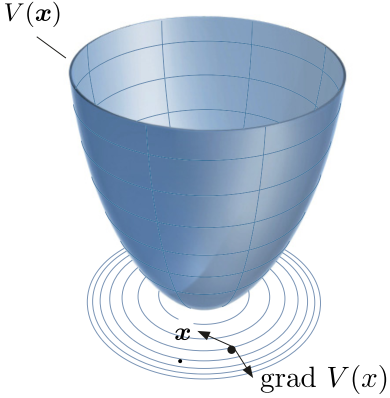




Conservation and Dissipation

Conservation of energy: $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) = 0$, i.e., the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Dissipation of energy: $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) \leq 0$, i.e., the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $(V(x) = c)$ make an obtuse angle.



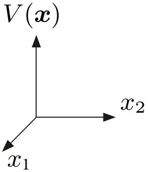


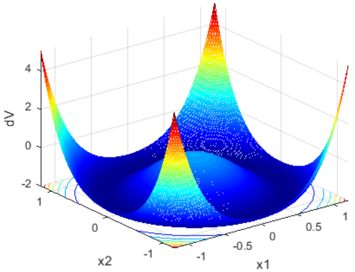
Illustration of the equation $\dot{V}(x) = \dot{x}^T \text{grad } (V(x)) < 0$

Lyapunov Stability Theorem

Not necessary to compute state trajectories: it is enough to check the sign of V and $\dot{V}$ in a neighborhood of the origin.

Example:

$$\begin{aligned} \dot{x}_1 &= x_1\left(x_1^2+x_2^2-2\right)-4x_1x_2^2 \\ \dot{x}_2 &= x_2\left(x_1^2+x_2^2-2\right)+4x_1^2x_2 \end{aligned}$$



Study the stability of the equilibrium state $\bar{x}=0$
Candidate Lyapunov function: $V(x)=x_1^2+x_2^2$
(positive definite in $\mathbb{R}^2$)

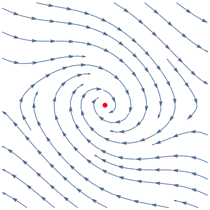
$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}f_1(x)+\frac{\partial V}{\partial x_2}f_2(x) \\ &= 2\left(x_1^2+x_2^2\right)\left(x_1^2+x_2^2-2\right) \end{aligned}$$

In the set level $x_1^2+x_2^2-2<0$ one has $\dot{V}$ is negative definite, therefore $\bar{x}=0$ is AS in the ball centered $B_{\sqrt{2}}(0)$.

Lyapunov Stability Theorem

The choice of the Lyapunov function is not unique.

Example 2: damped pendolum



$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - \sin(x_1)\end{aligned}$$

Study the stability of $\bar{x} = [0 \quad 0]^T$
Lyapunov candidate function: $V(x) = \underbrace{(1-\cos(x_1))}_{\text{potential en.}} + \underbrace{x_2^2/2}_{\text{kinetic en.}}$

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = \sin(x_1)x_2 + x_2(-x_2 - \sin(x_1)) = -x_2^2$$

$\dot{V}$ is negative semi definite in $\mathbb{R}^2$ (and then in $B_{2\pi(0)}$) $\Longrightarrow \bar{x} = 0$ is stable.

Physical intuition tells us the equilibrium is AS but the chosen Lyapunov function certifies only stability

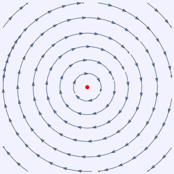
Lyapunov instability theorem

Example Study the stability of $\bar{x} = 0$.

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_1\left(x_1^2 + x_2^4\right) \\ \dot{x}_2 &= -2x_1 + x_2\left(x_1 + x_2^4\right) \end{aligned}$$

Linearized system around $\bar{x} = 0$

$$\begin{aligned} \dot{x}_1 &= 2x_2 \\ \dot{x}_2 &= -2x_1 \end{aligned} \qquad \text{eigenvalues}: \ \pm 2j$$



No conclusion on stability of (3).

Candidate Lyapunov function: $V(x) = (x_1^2 + x_2^2)/2$

$$\begin{aligned} \dot{V} &= x_2\left(2x_1 + x_1\left(x_1^2 + x_2^4\right)\right) + x_2\left(-2x_1 + x_2\left(x_1 + x_2^4\right)\right) \\ &= (x_1^2 + x_2^2)(x_1^2 + x_2^4) \rightarrow \text{positive definite in } \mathbb{R}^2 \end{aligned}$$

Then $\bar{x} = 0$ is unstable

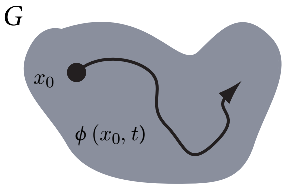
Invariance Principle: Krasovskii-Lasalle LaSalle's Theorem

LaSalle's invariance principle is a tool for assessing **asymptotic stability** properties of $\bar{x} = 0$ for $\dot{x} = f(x)$ when $\dot{V}(x)$ is only semi-definite

Review

A set $G \subseteq \mathbb{R}^n$ is (positively) invariant for $\dot{x} = f(x)$ if

$$x_0 \in G \Longrightarrow \phi(t, x_0) \in G, \quad \forall t \geq 0.$$



Invariance Principle: Local LaSalle theorem

Theorem: LaSalle's Invariance Principle

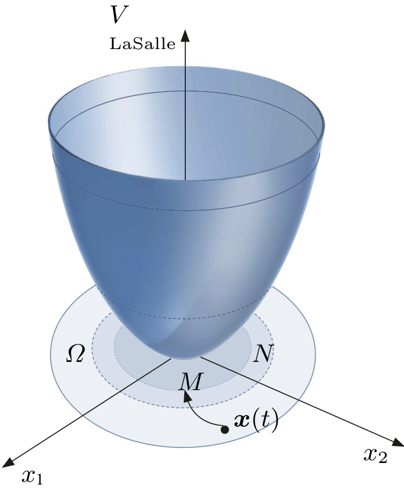
Let $\dot{x} = f(x)$ be a be a system with a compact positively invariant set Ω and let $V_{\text{LaSalle}}(x)$ be a continuously differentiable function with

$$\dot{V}_{\text{LaSalle}} \leq 0$$

for all $x \in \Omega$. Further, let N denote the set of all points $x \in \Omega$ with

$$\dot{V}_{\text{LaSalle}} = 0$$

and let M denote the largest invariant set in N. In this case all solutions $x(t)$ that start within Ω tend to the set M for $t \to \infty$



Invariance Principle: Local LaSalle theorem

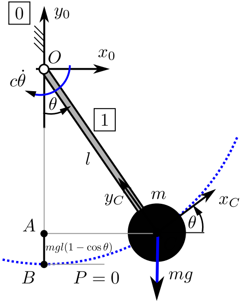
Remarks

- The theorem provides sufficient conditions for Ω to be a region of attraction for the set M
- Notable case: when $M = \{0\}$ the theorem gives a region of attraction (asymptotic stability) for the equilibrium state $\bar{x} = 0$.

Invariance Principle: Local LaSalle theorem

Example: Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{cases}$$



Consider the Lyapunov function

$$V(x) = \underbrace{\frac{g}{l}(1-\cos x_1)}_{\text{potential en.}} + \underbrace{\frac{1}{2}x_2^2}_{\text{kinetic en.}}$$

V(0)=0

V(x)>0 if $x_1 \in (-2\pi, 2\pi)$

Then V is positive definite in $B_2\pi(0)$

$$\dot{V}(x) = \frac{g}{l}\sin x_1 \dot{x}_1 + x_2\dot{x}_2 = -\frac{k}{m}x_2^2 \leq 0$$

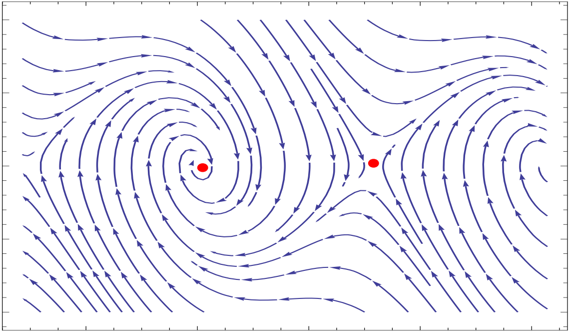
Then $S := \{(x_1, x_2) | x_2 = 0\}$, i.e., x_1 can be anything and S is the x_1-axis.

Invariance Principle: Local LaSalle theorem

Q. what is the largest invariant set ?

1. $x_2 \equiv 0 \Longrightarrow \dot{x}_2 \equiv 0$
2. $\dot{x}_2 = 0 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}.0 = -\frac{g}{l}\sin(x_1)$

$\sin(x_1) = 0$ must be satisfied. Locally, on the set of $x_1 \in (-\pi, \pi)$, this is only satisfied for $x_1 = 0$. Thus $\bar{x} = 0$ is locally asymptotically stable



Lyapunov theory for LTI systems

Let $x = \bar{x}$ be an equilibrium for the autonomous nonlinear system

$$\dot{x} = f(x) \tag{6}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a continuously differentiable function and $\mathcal{D}$ is a neighborhood of $\bar{x}$. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=\bar{x}}$$

Then:

$\bar{x}$ is **aymptotically stable** if $\mathrm{Re}(\lambda_i)<0$ for all eigenvalues of A.

$\bar{x}$ is **stable** if $\mathrm{Re}(\lambda_i)<0$ and $\mathrm{Re}(\lambda_i)=0$ for one of the eigenvalues of A.

$\bar{x}$ is **unstable** if $\mathrm{Re}(\lambda_i)>0$ for one or more of the eigenvalues of A.

In linear systems, *local stability* $\Longleftrightarrow$ *global stability.*

In nonlinear systems, this is not true.

Lyapunov theory for LTI systems

Review: Positive Definite Matrices

Symmetric matrix $M = M^T$ is

1. **positive definite** (pd) if $x^T M x > 0, \forall x \neq 0.$
2. **positive semi-definite** (psd) if $x^T M x \geq 0, \forall x \in \mathbb{R}^n.$

Lemma:

$$M = M^T > 0 \Longleftrightarrow \lambda_i(M) > 0$$
$$M = M^T \geq 0 \Longleftrightarrow \lambda_i(M) \geq 0$$

Properties of the quadratic function $x^T M x$

From (1) and (2) one has

if $M > 0$, $V(x) = x^T M x$ is a positive definite (pd) function.

if $M \geq 0$, $V(x) = x^T M x$ is a positive semi-definite (pd) function.

Lyapunov functions for LTI systems

Lyapunov functions for LTI systems

For linear system $\dot{x} = Ax$.
Consider as Lyapunov candidate function $V(x) = x^T P x$, $P = P^T > 0$.

$V(x)$ is quadratic, gpd and radially unbounded.

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x$

If $A^T P + PA < 0$, i.e. there is $Q > 0$ symmetric such that

$$A^T P + PA = -Q \tag{7}$$

then $\dot{V}$ is globally negative definite and by the second Lyapunov method the origin is globally asymptotically stable.

$A^T P + PA = -Q$ is called Lyapunov equation

Lyapunov theorem for LTI systems

Remarks

For LTI systems it is enough to consider quadratic Lyapunov functions.

Algorithm:

- choose $Q > 0$ (e.g. Q=I)
- solve $A^T P + PA = -Q$ (linear systems in the entries of the symmetric matrix P)
- The LTI system is AS if and only if $P > 0$

Example:

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}}_{A} x$$

Eigenvalues of $A : \{-1, -3\} \implies$ (global) asymptotic stability.

Choose $Q = Q^T = I_{2\times 2}$. Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, where $p_{12} = p_{21}$.

Lyapunov theorem for LTI systems

Solve the Lyapunov equation $A^T P + PA = -Q$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for p_{11}, p_{12} and p_{22} gives

$$\begin{cases} 2p_{11} & = & -1 \\ -4p_{12} + 4p_{11} & = & 0 \\ 8p_{12} - 6p_{22} & = & -1 \end{cases}$$

Solving the linear systems one gets

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

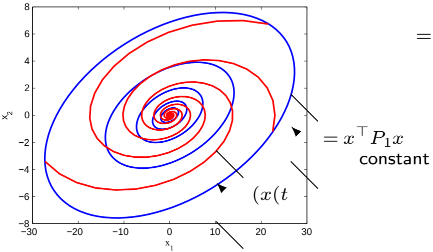
Since $P > 0 \Longrightarrow$ the systems is AS.

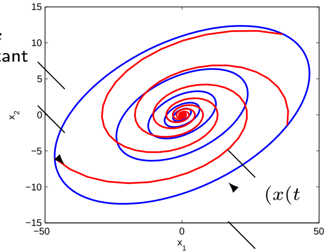
Example

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -16 \\ 1 & -2 \end{bmatrix}}_{A} x \quad \text{Eigenvalues of A} : \{-1 \pm j\sqrt{15}\}$$

Solve $PA + A^T P = -Q$ for P:

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$$





any choice of $Q > 0$ gives $P > 0$ (since A is strictly stable)

but not every $P > 0$ gives $Q > 0$